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Batalin-Vilkovisky quantization method with applications to gauge fixing

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Sommario

In una serie di articoli [2], [3] pubblicati tra il 1981 e il 1983, Igor Batalin e Grigory Vilkovisky svilupparono una procedura per quantizzare le teorie di gauge tramite un approccio basato sull'integrazione funzionale. Al giorno d'oggi questo è considerato il metodo più potente per la quantizzazione delle teorie di gauge. Lo scopo di questa tesi è l'applicazione del formalism BV ad alcune teorie di campo quantistiche topologiche di tipo Schwarz. E' presentata una formulazione BV della celebre teoria di Chern-Simons, la quale fu la prima teoria di campo quantistica topologica ad essere studiata da Witten nel suo famoso articolo del 1989 [30]. Di seguito viene presentata la cosidetta teoria di campo BF (probabilmente introdotta per la prima volta da Horowitz in [18]) su una varietà di dimensione arbitraria in una prospettiva BV. L'ultima applicazione che consideriamo è la formulazione BV del modello Sigma di Poisson introdotto da Cattaneo e Felder in [7]. In tutti questi modelli viene discussa dettagliatamente la procedura di gauge fixing.

Abstract

In a series of seminal papers [2], [3] written between the 1981 and 1983, Igor Batalin and Grigory Vilkovisky developed a procedure to quantize gauge theories via path integral approach. This algorithm nowadays is considered to be the most powerful quantization method for gauge theories. The aim of this thesis is the application of the BV formalism to some topological quantum field theories of Schwarz type. A BV formulation of the Chern-Simons theory, the celebrated topological quantum field theory first studied by E. Witten in his famous 1988 paper [30], is presented. Next, the so called BF field theory (Probably introduced for the first time by Horowitz in [18]) on manifolds of any dimension is studied in a BV perspective. The last topic we consider is the BV formulation of the Poisson sigma model introduced by Cattaneo and Felder in [7]. In all these model, we discuss in depth the implementation of the gauge fixing procedure.

Contents

Chapter 1

The Batalin-Vilkovisky quantization method

In this chapter we provide a summary history of the development of the Batalin-Vilkovisky formalism, then we discuss the method in detail introducing BV geometry, BV algebras and present the quantization scheme and the gauge fixing procedure.

1.1 The roots of the formalism

Fundamental interactions of nature are described by gauge theories. A gauge symmetry has a crucial role nowadays because signals that the related theory is described in a redundant way. In particular there are some degrees of freedom which do not enter in the lagrangian. A theory like that possesses local invariance. From a theoretical point of view we can eliminate this gauge degrees of freedom, but in practice, for many reasons (e.g. manifest covariance, locality of interactions or simply for calculation convenience), we do not do this. The gauge invariance problem was quoted for the first time by Richard P. Feynman in a conference held in a small town near Varsaw in 1962. In his talk titled "Quantum theory of gravitation", Feynman presented the problem of gauge invariance in Yang-Mills theory and in the gravitation. He proposed some heuristic methods to treat this question. In the following years other scientists introduced more sophisticated techniques to study these theories. There are many problems in the quantization methods of gauge theories. In the abelian case the procedure is in the most of the cases well understood. In contrast the situation is more complicated in the non-abelian case. In order to perform the quantization procedure we must introduce ghost fields. Therefore the gauge fixing method is necessary to render dynamical all the degrees of freedom, in this way the unitary is preserved. A further improvement about this topic was developed by L.D. Faddeev and V.Popov, and today is known as Faddeev-Popov method. It consists in an functional integral approach to the quantization in which the presence of auxiliary fields called ghosts is considered as a sort of measure effect. In fact, dividing out the volume of the gauge group, a Jacobian measure contribution arises. This factor is generated by introducing quadratic terms in the lagrangian for ghosts. The lagrangian with the gauge fixing contribute retains a global symmetry which was not broken by quantization. This property was discovered by Becchi,Rouet, Stora and Tyutin in 1974. A property of the BRST symmetry is that for closed theories the transformation law for the original fields has the same behavior of a gauge transformation where the gauge parameters are replaced by ghost fields. In order to develop a formulation of the gauge theories which contains ghost fields and incorporate the BRST symmetry was developed the field-antifield formulation. The advantage of this formalism is to consider the previous symmetry as a fundamental principle and use sources to deal with it. In 1975 J. Zinn-Justin studied the problem of the renormalization in the Yang-Mills theories. He introduced the sources for the BRST transformations and a symplectic structure denoted by $(\cdot,\cdot)$ in the space of fields and sources. Thanks to his idea he wrote the Slavnov-Taylor identity in the following compact form

$$(Z,Z)=0, \tag{1.1}$$

where Z is the generating functional of one-particle-irreducible diagrams. Contemporary the proof of the renormalizability of the gauge theories using the Feynman rules was proposed by t'Hooft and Veltman. Parallel to the development of the lagrangian formalism was developed also the hamiltonian formalism, which gained a lot of importance after the discovery of the BRST symmetry. A group of physicists including Igor Batalin, E.S Frakdin and Grigorij Vilkovisky studied the problem of the phase space integral quantization of the gauge theories. This problem was solved in 1977 for closed algebras, and thanks to Frakdin and Fradkina for the open algebras. Finally, we arrived to the Batalin-Vilkovisky formalism that was developed in many seminal series of papers written between 1981 and 1983. These physicists further developed the Zinn-Justin approach, generalizing the symplectic structure and the sources for the BRST transformation. They called them antibracket and antifield respectively. Due to their contributions this quantization procedure is called Batalin-Vilkovisky formalism (or BV for short). Nowadays is recognized as the most powerful method to treat gauge theories. The geometrical aspects of the BV formalism were studied later by Schwarz in [29]

1.2 BV geometry

In this section we present the geometry of Batalin-Vilkovisky quantization algorithm. let U be a domain of a graded manifold M parametrized with coordinates $(x^1,...x^n,\xi^1...\xi^n)$. Let F and G be functions on U, then we define a degree -1 Poisson bracket as follows

$$\{F,G\}=\frac{\partial_R F}{\partial x^i}\frac{\partial_L G}{\partial \xi_i}-\frac{\partial_R F}{\partial \xi_i}\frac{\partial_L G}{\partial x^i}, \tag{1.2}$$

where ∂_R and ∂_L denote the right derivative and the left derivative respectively. We define the transformations of the domain of a superspace preserving the bracket (1.2) as P-transformations. They are an odd version of the symplectic transformations. Furthermore, if we impose the condition that the Jacobian is equal to one, we obtain the so called SP-transformations.

The graded manifold M is equipped with a degree -1 symplectic form. It has, in a general local coordinate system $(z^1,..,z^{2n})$, the following expression

$$\omega=\frac{1}{2}dz^i\omega_{ij}dz^j, \tag{1.3}$$

and is closed as expected, i.e

$$d\omega = 0. \tag{1.4}$$

Using the invertible matrix $\omega_{ij}(z)$, which inverse determines (1.3), we can express bracket (1.2) as follows

$$\{F, G\} = \frac{\partial_R F}{\partial z^i} \omega^{ij}(z) \frac{\partial_L G}{\partial z^j}, \tag{1.5}$$

where as in (1.2), F and G are defined on the graded space of functions.

We introduce some useful definitions:

Definition 1.1 A P-manifold is a standard supermanifold M pasted together from $(n|n)$-dimensional superdomanins by means of P-transformations.

Definition 1.2 A SP-manifold is a standard supermanifold M pasted together from $(n|n)$-dimensional superdomanins by means of SP-transformations.

Using an important result of the standard symplectic geometry we can construct a unique vector field K_H (Hamiltonian vector field) corresponding to a function H (hamiltonian) on a P-manifold M by the following equation

$$K_H^i = \omega^{ij}(z) \frac{\partial_L H}{\partial z^j}. \tag{1.6}$$

If the function H is odd, then K_H is even and viceversa.
We can provide now an invariant form for definition 1.1 and 1.2.

Definition 1.3 A P-manifold is a graded manifold M endowed by a non-degenerate degree -1 closed 2-form ω.

Definition 1.4 A SP-manifold is a graded manifold M endowed by non-degenerate closed degree -1 2-form ω and by a density function $\rho(z)$.

There are three important remarks about the previous definitions:

Remark 1.1 Definition 1.1 is equivalent to definition 1.3 as consequence of Darboux's theorem, which states that a non degenerate closed odd 2-form ω can be locally written as follows

$$\omega = dx^i d\xi_i, \tag{1.7}$$

using an appropriate choice of coordinates $(x^1...x^n, \xi_1, ..., \xi_n)$- Darboux coordinates. (1.7) is the coordinate berezinian of the all Darboux charts of an atlas of the manifold M.

Remark 1.2 In the definition (1.4) there is the density function $\rho(z)$. It is not arbitrary. We require that in the neighbourhood of every point in M, we can choice appropriately the Darboux coordinates such that $\rho(z) = 1$.

Remark 1.3 There is not a classical analogous of P-manifold and SP-manifold because always symplectic matrix has a determinant equal to one.

We consider a SP-manifold with a berezinian of the form (1.7). We can introduce the following degree 1 second order differential operator

$$\Delta : C^\infty(M) \longrightarrow C^\infty(M), \tag{1.8}$$

which is the Batalin-Vilkovisky laplacian. Locally we can express it in Darboux charts as

$$\Delta = \int_M \sum_i \frac{\partial_R}{\partial x^i}\frac{\partial_L}{\partial \xi_i} \tag{1.9}$$

BV laplacian is a degree +1 odd second order operator and is nilpotent, i.e

$$\Delta^2 = 0 \tag{1.10}$$

Proof of relation (1.10)

$$\begin{aligned}\Delta^2 &= \sum_{ij} \frac{\partial_R}{\partial x^i}\frac{\partial_L}{\partial \xi_i}\frac{\partial_R}{\partial x^j}\frac{\partial_L}{\partial \xi_j} \\ &= \sum_{ij}(-1)^{|x^i|\ |\xi_j|+|x^i|\ |x^j|+|\xi_i|\ |\xi_j|+|\xi_i|\ |x^j|}\frac{\partial_R}{\partial x^j}\frac{\partial_L}{\partial \xi_j}\frac{\partial_R}{\partial x^i}\frac{\partial_L}{\partial \xi_i} \\ &= \sum_{ij}(-1)^{(|x^i|+|\xi_i|)(|x^j|+|\xi_j|)}\frac{\partial_R}{\partial x^j}\frac{\partial_L}{\partial \xi_j}\frac{\partial_R}{\partial x^i}\frac{\partial_L}{\partial \xi_i}\end{aligned} \tag{1.11}$$

Using the property that x^i and ξ_i have opposite parity, we have $|x^i| + |\xi_i| = -1$, for any i. Indeed we obtain

$$\Delta^2 = -\Delta^2, \tag{1.12}$$

then, we get relation (1.10). □

We can introduce the BV laplacian in another way. Consider a P-manifold with μ a general berezinian of the manifold M. We can define the second order operator

$$\Delta_\mu : C^\infty(M) \longrightarrow C^\infty(M), \tag{1.13}$$

by setting

$$\Delta_\mu(H) := \frac{1}{2} div_\mu K_H, \tag{1.14}$$

where div_μ is a degree 0 first order differential operator uniquely defined by the following property

$$\int_M \mu K_H F = -\int_M \mu\ div_\mu K_H F, \tag{1.15}$$

and K_H is the vector field introduced in (1.6).
Locally in a Darboux chart (x^i, ξ_i) on the manifold M, we assume that the berezinian has a local form

$$\mu = \rho(x,\xi) dx^1 ... dx^n d\xi_n ... d\xi_1, \tag{1.16}$$

where $\rho(x, \xi)$ is a local density function.
Imposing the condition $\rho(x, \xi) = 1$, (1.16) has the following form

$$\mu = dx^1 ... dx^n d\xi_n ... d\xi_1 \tag{1.17}$$

We can formulate (1.14) as follows

$$\Delta_\mu(H) = \sum_i \frac{\partial_R}{\partial x^i} \frac{\partial_L}{\partial \xi_i} H + \frac{1}{2} \{log\ \rho, H\} \tag{1.18}$$

The BV laplacian in equation (1.18) is not necessary nilpotent. It squares to zero if and only if the Berezinian has the form (1.17). In this case (1.18) is equivalent to (1.9).

Lagrangian Submanifold

We introduce now the relevant notion of lagrangian submanifold

Definition1.5 Let M a graded manifold and ω a degree -1 symplectic form. We can define the natural injection $i_{\mathscr{L}}$ as the following application

$$i_{\mathscr{L}} : \mathscr{L} \longrightarrow M, \tag{1.19}$$

where $\mathscr{L}$ is a submanifold of M. $\mathscr{L}$ is a lagragian submanifold if

$$i_{\mathscr{L}}{}^* \omega = 0. \tag{1.20}$$

There exists a berezinian $\mu|_{\mathscr{L}}{}^{\frac{1}{2}}$ on $\mathscr{L}$, which is a tensor square root of the restriction of μ to $\mathscr{L}$.

Definition 1.6 Let M a graded manifold with a berezinian (1.17) and a degree -1 symplectic form ω, we define a BV integral as follows

$$\int_{\mathscr{L} \subset M} F \sqrt{\mu|_{\mathscr{L}}}, \tag{1.21}$$

with $\mathscr{L} \subset M$ is the lagrangian submanifold and $F \in C^\infty(M)$ is a function satisfying $\Delta_\mu F = 0$.

We present now the BV version of the Stokes' theorem:

Theorem 1.1 (Batalin-Vilkovisky-Schwarz) Let M be a graded manifold endowed with a Berezinian (1.17) and and a degree -1 symplectic form ω.

(i) For any $G \in C^\infty(M)$ and $\mathscr{L} \subset M$ is a lagrangian submanifold, we obtain

$$\int_{\mathscr{L}} \Delta_\mu G \sqrt{\mu|_{\mathscr{L}}} = 0 \tag{1.22}$$

(ii) Let $\mathscr{L}$ and $\mathscr{L}'$ lagrangian submanifolds of a SP manifold M, whose are in the same homology class. Therefore

$$\int_{\mathscr{L}} F\sqrt{\mu|_{\mathscr{L}}} = \int_{\mathscr{L}'} F\sqrt{\mu|_{\mathscr{L}'}}, \tag{1.23}$$

where $F \in C^\infty(M)$ is a function satisfying $\Delta_\mu F = 0$.

1.3 BV algebras

The aim of this section is introducing the BV algebras and present some of their properties.

Definition 1.7 Let V a commutative graded algebra. V is called a Gerstenhaber algebra if is equipped with bilinear bracket of the form (1.2) with the following properties:

$$\{F, G\} + (-1)^{(|F|+1)(|G|+1)}\{G, F\} = 0 \tag{1.24}$$

$$\{F, \{G, H\}\} = \{\{F, G\}, H\} + (-1)^{(|F|+1)(|G|+1)}\{G, \{F, H\}\} \tag{1.25}$$

$$\{F, GH\} = \{F, G\}H + (-1)^{(|F|+1)|G|}G\{F, H\} \tag{1.26}$$

$$\{FG, H\} = F\{G, H\} + (-1)^{|G|(|H|+1)}\{F, H\}G \tag{1.27}$$

where $F, G, H \in V$.
The degree of the bracket $\{\cdot, \cdot\}$ is 1. We can verify the properties (1.24)-(1.25)-(1.26)-(1.27) using the definition (1.2). An example of Gerstenhaber algebra is $C^\infty(T^*[-1]N)$.

Definition 1.8 A BV algebra is a Gerstenhaber algebra equipped with a degree +1 linear map

$$\Delta : V \longrightarrow V \tag{1.28}$$

which is nilpotent and generates the bracket

$$\{F, G\} = (-1)^{|F|}\Delta(FG) + (-1)^{(|F|+1)}(\Delta F)G - F\Delta G. \tag{1.29}$$

Where (1.28) is called BV laplacian or odd laplacian.
The space of functions $C^\infty(T^*[-1]N)$ is a BV algebra V with Δ defined in (1.18), with a choice of the volume form.

Definition 1.9 A BV manifold is a graded manifold such that the space of functions $C^\infty(M)$ is equipped with the structure of a BV algebra.
For example $T^*[-1]N$ is a BV manifold.

We can provide the definition of a BV algebra proposed by E. Getzler (see [12]) in 1994 which is equivalent to the previous one.

Definition 1.10 A BV algebra V is a Gerstenhaber algebra equipped with a degree +1 linear map

$$\Delta : V \longrightarrow V, \tag{1.30}$$

that is nilpotent and satisfies the following relation

$$\begin{aligned}\Delta(FGH) = \Delta(FG)H + (-1)^{|F|}F\Delta(GH) + (-1)^{(|F|-1)|G|}G\Delta(FH) +\\ - \Delta(F)GH - (-1)^{|F|}F\Delta(G)H - (-1)^{|F|+|G|}FG\Delta H.\end{aligned} \tag{1.31}$$

BV bracket generalizes the Schouten bracket defined on polyvector fields, viz. controvariant tensor.

1.4 BV quantum master equation and gauge fixing procedure

In this section we discuss the gauge fixing procedure, then we obtain the BV quantum master equation.

Consider now the following integral

$$\int_N \Phi \sqrt{\mu} \tag{1.32}$$

where μ is the berezinian (1.17) and $N = T^*[-1]\mathscr{F}$ is the space of fields and antifields. We assume that Φ has the following form

$$\Phi = Xe^{\frac{i}{\hbar}S}, \tag{1.33}$$

where S is the quantum action.
We need a lagrangian submanifold $\mathscr{L}$ in order to use the definition of BV integral (1.21). To choose $\mathscr{L}$ we use the gauge fixing procedure. The gauge fixing paves the way to the quantization of the gauge theories via the path integral approach, and the most important data is the gauge fixing fermion, an odd functional of fields with ghost number -1. We can exemplify the importance of the gauge fixing in the following way. Consider a model which is described by some classical fields $\varphi^{(0)i}$ (in a 4d gauge theory are the usual gauge field A_μ). Now we introduce ghosts c in order to obtain the fields φ^i. The action of the model $S[\varphi^i]$ is ill-defined, so we need a new ones to quantize the theory using the path integral technique. To do this we use the BV formalism. We add antifields φ^*_i and we have a BV action $S[\varphi^i, \varphi^*_i]$ for the model, which possesses a gauge invariance, so we cannot use it to implement the path integral in equation (1.32). We can set the antifields to zero, but in this way, we reduce the action to the classical one that cannot be used to perform a quantization procedure because is ill-defined. Usually we eliminate the antifields by using a gauge fixing fermion Ψ via

$$\varphi^*_i = \frac{\partial \Psi}{\partial \varphi^i}, \tag{1.34}$$

where φ^i and φ^*_i denote the fields and antifields respectively. The gauge fixing fermion ψ is a degree -1 functional which depends only by the fields.
Thanks to (1.34) we select a lagrangian submanifold $\mathscr{L}$

$$\mathscr{L} \longleftrightarrow \left\{ \varphi^*_i = \frac{\partial \Psi}{\partial \varphi^i} \right\} \tag{1.35}$$

We denote the gauge fixed action as $S|_{\mathscr{L}} = S_{\mathscr{L}}$.

We can check that (1.35) represents a lagrangian submanifold

$$d\varphi^i \wedge d\varphi^*_i = d\varphi^i \wedge d\varphi^j \frac{\partial^2 \Psi}{\partial \varphi^i \partial \varphi^j} = 0$$

where we used the property that a contraction of a symmetric tensor with an anti-symmetric tensor vanishes.

Now we are interested in this BV integral

$$\int_{\mathscr{L} \subset M} \Phi \, \sqrt{\mu|_{\mathscr{L}}}, \tag{1.36}$$

where $\sqrt{\mu|_{\mathscr{L}}}$ is the berezinian restricted to the lagrangian submanifold $\mathscr{L}$ Selecting a suitable Ψ for a given model is a matter of skill, does not exist an algorithm or a theorem that states how to choose a gauge fermion.
There are two important remarks about the gauge fixing procedure

Remark 1.4 Usually we choose the gauge fermion Ψ in order to have a non degenerate theory. It means that when the action is expanded about a solution of the equation of motion, propagators exist. Such Ψ is called admissible.

Remark 1.5 When the classical action S_0 is local is preferable that the gauge fermion be a local functional of the fields, in order to preserve the locality of the gauge-fixed action.

Now we obtain the BV quantum master equation.
To discuss a quantum theory we construct a path integral which contains constraint (1.35) in a delta function form as follows

$$Z_\Psi(X) = \int d\varphi d\varphi^* \delta \left(\varphi^*_i - \frac{\delta \Psi}{\delta \varphi^i} \right) exp \left(\frac{i}{\hbar} S[\varphi, \varphi^*] \right) X(\varphi, \varphi^*), \tag{1.37}$$

where $X[\varphi, \varphi*]$ is a correlation function and $S[\varphi, \varphi^*]$ is the quantum action which depends from fields and antifields. We denote the integrand in (1.37) by $\mathfrak{X}[\varphi, \varphi^*]$.
The freedom in the choice of the parameter Ψ corresponds to the gauge fixing procedure. The final result obtained by the model should be independent from the gauge fixing, so we determine under which conditions this happens. Consider a deformation of the gauge fermion of an infinitesimal quantity, then

$$\begin{aligned} & Z_{\Psi+\delta\Psi}(X) - Z_\Psi(X) = \\ & \int d\varphi \left(\mathfrak{X} \left[\varphi, \frac{\partial \Psi}{\partial \varphi} + \frac{\partial \delta \Psi}{\partial \varphi} \right] - \mathfrak{X} \left[\varphi, \frac{\partial \Psi}{\partial \varphi} \right] \right) = \\ & = \int d\varphi \frac{\partial_R \mathfrak{X}}{\partial \varphi^*} \frac{\partial_L \delta \Psi}{\partial \varphi} + O\left((\delta \Psi)^2 \right), \end{aligned}$$

after an integration by parts we have

$$\int d\varphi \Delta \mathfrak{X} \delta \Psi + O((\delta \Psi)^2).$$

According to the previous equation, the integral (1.37) is infinitesimally independent of Ψ iff

$$\Delta \mathscr{L} = 0. \tag{1.38}$$

We have the following theorem

Theorem 1.2 Let the function Φ as in (1.33) with $X = 1$, we have the following quantum master equation

$$\{S, S\}_{BV} - 2i\hbar\Delta S = 0, \tag{1.39}$$

where the BV bracket has the form

$$\{S, S\}_{BV} = \Delta(S^2) - S\Delta S - (\Delta S)S \tag{1.40}$$

Sketch of the proof of theorem 1.2
Using (1.33) in (1.38) we have

$$\Delta\left(exp\left\{\frac{i}{\hbar}S\right\}\right) = \Delta\left(\sum_{n=0}^{\infty}\frac{1}{n!}\frac{i^n}{\hbar^n}S^n\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{i^n}{\hbar^n}\Delta S^n \tag{1.41}$$

We must evaluate (1.41) for different n in order to find a relation for a general expression. In this calculation we repetitively use the following relation

$$\Delta(S^2) = (\Delta S)S + S\Delta S + \{S, S\}_{BV}. \tag{1.42}$$

Therefore, for a generic n, the following relation holds true

$$\frac{i^n}{\hbar^n}\left[nS^{n-1}\Delta S + \frac{1}{2}n(n-1)S^{n-2}\{S, S\}_{BV}\right]. \tag{1.43}$$

Now substituting (1.43) in (1.41) and after splitting the sum, (1.39) holds true. □

theorem 1.3 Let Φ as in (1.33), now with a non vanishing X, so we have

$$\{S, X\}_{BV} - i\hbar\Delta X = 0, \tag{1.44}$$

where S is the BV action which satisfies (1.39).

Proof of theorem 1.3.
Using (1.33) in (1.38), we have

$$\begin{aligned} 0 = \Delta\left(Xe^{\frac{i}{\hbar}S}\right) &= \Delta e^{\frac{i}{\hbar}S}\, X + e^{\frac{i}{\hbar}S}\Delta X + \left\{e^{\frac{i}{\hbar}S}, X\right\}_{BV} = \\ &= e^{\frac{i}{\hbar}S}\left(\Delta X + \frac{i}{\hbar}\{S, X\}_{BV}\right), \end{aligned}$$

then, we obtain

$$\Delta X + \frac{i}{\hbar}\{S, X\}_{BV} = 0, \tag{1.45}$$

which is another form of (1.44). □

Chapter 2

Chern-Simons Theory

In this chapter we present the application of the BV formalism to some models of topological field theory. In the first section we provide some fundamental notions of topological field theory and related properties, then we present the ordinary Chern-Simons theory and using the BV algorithm we quantize it.

2.1 Elements of topological quantum field theory

we define a field theory to be topological if it contains a set of operators $\mathcal{O}_i$ called observables whose correlation functions do not depend on the metric chosen on M. Formally we have:

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{i_1} ... \mathcal{O}_{i_n} \rangle = 0, \tag{2.1}$$

where $g^{\mu\nu}$ is the metric defined on M and $\mathcal{O}_i$ are the observables of the theory.
Topological field theories are very important because are non trivial field theories which they are not renormalizable, and so in some cases are completely solvable. Renormalization is not necessary because a topological theory is metric independent, therefore the ideas of points extremely closed or extremely far, from which ultraviolet and infrared divergences come, lost their meaning.
Topological field theories are very important not only for physics, but they are topological relevant in mathematics, because they provide an analytic expression for topological invariants in low dimensions.
We can group topological field theories in two classes:

I **Schwarz type**

II **Cohomological type** (or Witten type)

We describe the main differences briefly:

I **Schwarz theories** are gauge theories with a metric indipendent classical action. In this case the gauge fixing procedure is necessary in order to provide the quantum theory.

II **Cohomological theories** are supersymmetric theories which are not manifestly metric indipendent at classical level. They have a nilpotent odd operator denoted as Q. Physical observables are Q-cohomology classes and amplitudes involving these observables are metric independent because of decoupling of BRST trivial degrees of freedom. An example of such theories is the Donalds-Witten theory, which is the twisted $N = 2$ supersymmetric version of the Yang-Mills theory

In this thesis only topological quantum field theory of Schwarz type are considered. First we analyze the Chern-Simons theory from an ordinary and a BV perspective.

2.2 Ordinary Chern-Simons theory

2.2.1 Introduction

As a first example of Schwarz type topological field theory we present the Chern-Simons theory. This model gained its popularity with the famous paper written by Witten [30] in 1989. Witten understood that the Chern-Simons theory is quantizable and solvable in the appropriate sense. Indeed he showed the connection with topology an knot theory.

2.2.2 Classical action and Symmetries

First of all we introduce the geometrical framework we need to describe the Chern-Simons theory.
Consider a principal $G-$bundle P on a 3-fold M. G is a compact semisimple Lie group. We assume a trivial principal bundle, $P = M \times G$.
The dynamic fields are connections $A \in \Omega^1(M, \mathfrak{g})$, where $\mathfrak{g}$ is the gauge Lie algebra of the group G.
The Chern-Simons action for A is defined as follows

$$S_{CS}(A) = \frac{k}{4\pi} \int_M \left(A, dA + \frac{1}{3} [A, A] \right), \tag{2.2}$$

where $k \in \mathbb{R}$ is the coupling constant and $(\cdot, \cdot)$ is an invariant bilinear form on $\mathfrak{g}$. If we consider this form to be realized by the trace over some representation of the Lie algebra $\mathfrak{g}$, we can write the action in the following well-known form

$$S_{CS}(A) = \frac{k}{4\pi} \int_M Tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{2.3}$$

The action (2.3) is topological because it does not depend on a choice of a metric on M.
Let is calculated the equation of motion. From the Euler-Lagrange equation one finds

$$\frac{\delta S_{CS}(A)}{\delta A} = \frac{k}{2\pi} F_A = 0 \tag{2.4}$$

Equation (2.4) corresponds to the condition of flatness for the connection A.
Now we discuss the integration over the base manifold M. The Chern-Simons action is invariant under gauge transformation of the gauge field A. This invariance does not hold completely.

Consider indeed a gauge transformation $g \in Map(M, G)$, where $Map(M, G)$ is the group of continuous maps $M \longrightarrow G$ acting on the connection A as

$$A \longrightarrow A' = gAg^{-1} - dgg^{-1} \tag{2.5}$$

The Chern-Simons lagrangian varies as

$$\mathscr{L}_{CS}(A') = \mathscr{L}_{CS}(A) - \frac{k}{12\pi} Tr(g^{-1}dgg^{-1}dgg^{-1}dg) - \frac{k}{4\pi} dTr(Ag^{-1}dg) \tag{2.6}$$

The last term in equation (2.6) is exact, indeed can be neglected when we integrate on the compact manifold M, therefore does not affect the action. There are no reasons for the second term of equation (2.6) to vanish somehow, and this leads to a gauge non-invariance of the theory. Note that the second term can be scaled and in this way the invariance holds only up to integers. This determines that the action $S_{CS}(A)$ can be defined as a functional taking $\mathbb{R}/\mathbb{Z}$ or simply $\mathbb{R}$ values. With a suitable choice of k it is possible to enforce that the variation of the action due to a gauge transformation is $2\pi k$, with $k \in \mathbb{Z}$. Therefore, the fundamental quantity

$$e^{iS_{CS}(A)} \tag{2.7}$$

will be well defined, leading to a sensible quantum theory.
Consider the following integral

$$w(g) = \frac{1}{24\pi^2} \int_M Tr(g^{-1}dgg^{-1}dgg^{-1}dg), \tag{2.8}$$

for $g : M \longrightarrow G$ is called the winding number of the map g in topology.
It is a classical result that $w(g)$ is an integer.
We can restrict k to be

$$\mathscr{K} = \frac{k}{4\pi}, \tag{2.9}$$

with $k \in \mathbb{Z}$. To restrict the gauge anomaly of the Chern-Simons action generated by the term

$$\frac{\mathscr{K}}{3} Tr(g^{-1}dgg^{-1}dgg^{-1}dg) \tag{2.10}$$

to 2π times on integer k remains a free parameter which is called the level of the theory.

2.3 BV Chern-Simons theory

2.3.1 Geometrical framework and superfield formalism

We apply the BV formalism to the Chern-Simons theory in three dimensions. In this model the geometrical framework is the following:

(i) An oriented smooth compact 3-fold M

(ii) A principal $G-$bundle P over M. Here G is a compact Lie group

In this model we adopt the superfield formalism. The base of this theory is the degree +1 shifted tangent bundle of M, $T[1]M$.
In this case the related bundle projection is

$$\Pi : T[1]M \longrightarrow M \tag{2.11}$$

We consider now the degree +1 shifted adjoint bundle of P, $AdP[1]$.
One has $AdP[1] = M_{\times_G}\mathfrak{g}[1]$, where $\mathfrak{g}[1]$ is the degree +1 shifted Lie algebra of G. $AdP[1]$ is a vector bundle over M, then $\Pi^* AdP[1]$ is a vector bundle over $T[1]M$.
The superfield content of this model consists of the following superfield

$$\underline{a} \in \Gamma\left(T[1]M, \Pi^* AdP[1]\right). \tag{2.12}$$

$\underline{a}$ can be decomposed in homogeneous components of defined $T[1]M$ and $AdP[1]$ degree, called form and ghost degree respectively, yielding the form-ghost bidegree

$$\underline{a} = -c + a + a^\dagger - c^\dagger, \tag{2.13}$$

where:

$$c \qquad (0,1) \tag{2.14}$$

$$a \qquad (1,0) \tag{2.15}$$

$$a^\dagger \qquad (2,-1) \tag{2.16}$$

$$c^\dagger \qquad (3,-2) \tag{2.17}$$

The superfield $\underline{a}$ can be decomposed as follows

$$\underline{A} = \underline{A_0} + \underline{a}. \tag{2.18}$$

Where $\underline{A_0}$ is an ordinary background connection of P viewed as a locally defined field of form-ghost bidigree $(1,0)$.

We discuss now the integration of superfields. We can perform this integration using the standard supermeasure μ of $T[1]M$, μ has $T[1]M$ degree -3.
If φ is a superfield, one has:

$$\int_{T[1]M} \mu\varphi = \int_M \varphi^{(3)}, \tag{2.19}$$

where $\varphi^{(3)}$ is the component of φ of $T[1]M$ which has standard form degree 3.

BV symplectic form

This theory it is characterized by a symplectic form which is relevant in this description

$$\Omega_{BV} = \frac{1}{2}\int_{T[1]M} \mu Tr(\delta\underline{A}\delta\underline{A}), \tag{2.20}$$

using relation (2.18), we have

$$\frac{1}{2}\int_{T[1]M} \mu Tr\left[\delta(\underline{A_0} + \underline{a})\delta(\underline{A_0} + \underline{a})\right] = \tag{2.21}$$

Since $\delta(A_0) = 0$, then

$$\Omega_{BV} = \frac{1}{2}\int_{T[1]M} \mu Tr(\delta\underline{a}\delta\underline{a}). \tag{2.22}$$

The symplectic form in the previous equation has degree -1 and is closed as required. We can try to workout a BV formulation of the model. The BV antibracket reads as follows

$$\{F, G\}_{BV} = \int_{T[1]M} \mu Tr\left(\frac{\delta_R F}{\delta\underline{a}}\frac{\delta_L G}{\delta\underline{a}}\right). \tag{2.23}$$

Using the equation (2.13), we can express (2.22) as

$$\Omega_{BV} = \int_M \mu Tr(\delta c^\dagger \delta c + \delta a^\dagger \delta a). \tag{2.24}$$

So we have

$$\{F, G\} = \int_M \mu Tr\left[\frac{\delta_R F}{\delta c}\frac{\delta_L G}{\delta c^\dagger} - \frac{\delta_R F}{\delta c^\dagger}\frac{\delta_L G}{\delta c} + \frac{\delta_R F}{\delta a}\frac{\delta_L G}{\delta a^\dagger} - \frac{\delta_R F}{\delta a^\dagger}\frac{\delta_L G}{\delta a}\right] \tag{2.25}$$

2.3.2 Chern-Simons BV action

The Chern-Simons action in a BV formulation is formally:

$$S_{BV} = k\int_{T[1]M} \mu Tr\left[\underline{AdA} + \frac{2}{3}\underline{AAA}\right]. \tag{2.26}$$

Using the decomposition (2.18), we formulate the BV Chern-Simons action as

$$S_{BV} = k\ S_{CS}(A_0) + k\int_{T[1]M} \mu Tr\left[2\underline{a}\ \underline{F}_{A_0} + \underline{a}D_{A_0}\underline{a} + \frac{2}{3}\underline{aaa}\right], \tag{2.27}$$

where $S_{CS}(A_0)$ is the Chern-Simons action of A_0

$$S_{CS}(A_0) = k\int_M Tr\left(A_0 dA_0 + \frac{2}{3}A_0A_0A_0\right). \tag{2.28}$$

Proof of relation (2.27)
First of all consider equation (2.26) and use relation (2.18)

$$\begin{aligned}
&Tr\left(\underline{AdA} + \frac{2}{3}\underline{AAA}\right) = \\
&= Tr\Bigg((\underline{A_0} + \underline{a})\underline{d}(\underline{A_0} + \underline{a}) + \frac{2}{3}\left\{(\underline{A_0} + \underline{a})(\underline{A_0} + \underline{a})(\underline{A_0} + \underline{a})\right\}\Bigg) = \\
&= Tr\Big(\underline{A_0}\underline{d}\underline{A_0} + \underline{A_0}\underline{da} + \underline{ad}\underline{A_0} + \underline{ada} + \\
&+ \frac{2}{3}\left\{\underline{A_0A_0A_0} + 3\underline{A_0}^2\underline{a} + 3\underline{a}^2\underline{A_0} + \underline{aaa}\right\}\Big) =
\end{aligned}$$

$$= Tr\Big(\underline{A_0}\underline{d}\underline{A_0} + \frac{2}{3}\underline{A_0 A_0 A_0} + \underline{a}(\underline{d}\underline{A_0} + 2\underline{A_0}^2) +$$

$$+ \underline{A_0}\underline{da} + \underline{a}(\underline{da} + 2\underline{a}\underline{A_0}) + \frac{2}{3}\underline{aaa}\Big) =$$

$$= Tr\Bigg(\underline{A_0}\underline{d}\underline{A_0} + \frac{2}{3}\underline{A_0 A_0 A_0} + \underline{a}F_{\underline{A_0}} + \underline{a}\underline{A_0}^2 - d(\underline{A_0}\underline{a}) +$$

$$+ \underline{ad}\underline{A_0} + \underline{a}(\underline{da} + \underline{A_0}\underline{a} + \underline{a}\underline{A_0}) + \frac{2}{3}\underline{aaa}\Bigg) =$$

Integrating the previous lagrangian over the graded manifold $T[1]M$ one gets

$$\begin{aligned} &= \int_{T[1]M} \mu Tr\left[\underline{A_0}\underline{d}\underline{A_0} + \frac{2}{3}\underline{A_0 A_0 A_0} - d(\underline{A_0}\underline{a})\right] + \\ &+ \int_{T[1]M} \mu Tr\left[2\underline{a}F_{\underline{A_0}} + \underline{aD}_{\underline{A_0}}\underline{a} + \frac{2}{3}\underline{aaa}\right]. \end{aligned} \tag{2.29}$$

Consider now the first integral in the equation (2.29) and using a sort of Stokes' theorem we can eliminate the term $d(\underline{A_0}\underline{a})$, which is not globally defined

$$\begin{aligned} &\int_{T[1]M} \mu Tr\left[\underline{A_0}\underline{d}\underline{A_0} + \frac{2}{3}\underline{A_0 A_0 A_0} - \underline{d}(\underline{A_0}\underline{a})\right] = \\ &= \int_{M} \mu Tr\left[A_0 dA_0 + \frac{2}{3}A_0 A_0 A_0\right], \end{aligned} \tag{2.30}$$

that corresponds to equation (2.28). □

The integrand in equation (2.28) is not globally defined, and so the integration is to be understood in the Cheeger-Simons character sense.

Leaving aside the problem to giving the meaning of (2.30), we quantize the theory with BV algorithm.

2.3.3 Classical BV equation

We can demonstrate the classical BV equation, namely

$$\{S_{BV}, S_{BV}\}_{BV} = 0 \tag{2.31}$$

Proof of relation (2.31).

In order to demonstrate the above relation we calculate the following directional derivatives

$$\frac{d}{dt}S_{BV}(\underline{a} + t\underline{b})|_{t=0} = \frac{d}{dt}\left\{kS_{CS}(A_0) + kS_{CS}(\underline{a} + t\underline{b})\right\} \tag{2.32}$$

Since the fact $k\frac{d}{dt}S_{CS}(A_0) = 0$, we have

$$\begin{aligned} \frac{d}{dt}k\int_{T[1]M} \mu Tr\Big[&2(\underline{a} + t\underline{b})\underline{F}_{\underline{A_0}} + (\underline{a} + t\underline{b})\underline{D}_{\underline{A_0}}(\underline{a} + t\underline{b}) + \\ &+ \frac{2}{3}(\underline{a} + t\underline{b})(\underline{a} + t\underline{b})(\underline{a} + t\underline{b})\Big] = \end{aligned}$$

$$= k\int_{T[1]M} \mu Tr\left\{2\underline{b}\underline{F_{A_0}} + \underline{aD}_{A_0}\underline{b} + \underline{bD}_{A_0}\underline{a} + 2\underline{aab}\right\} =$$

we can rewrite the above relation as follows

$$
\begin{aligned}
& k \int_{T[1]M} \mu Tr \left\{ 2\underline{b}\underline{F_{A_0}} + \underline{a}\underline{D}_{A_0}\underline{b} + \underline{D}_{A_0}\underline{ab} - \underline{D}_{A_0}(\underline{ab}) + 2\underline{aab} \right\} = \\
& = 2k \int_{T[1]M} \mu Tr \left\{ \underline{b}F_{\underline{A_0}} + \underline{b}\underline{D}_{\underline{A_0}}\underline{a} + \underline{baa} \right\} = \\
& = 2k \int_{T[1]M} \mu Tr \left\{ \underline{b} \left(F_{\underline{A_0}} + \underline{D}_{\underline{A_0}} + \underline{aa} \right) \right\} .
\end{aligned}
$$

Hence

$$
\frac{\delta_L S_{BV}}{\delta \underline{a}} = \frac{\delta_R S_{BV}}{\delta \underline{a}} = 2k \left(F_{\underline{A_0}} + \underline{D}_{A_0}\underline{a} + \underline{aa} \right) \tag{2.33}
$$

From equations (2.23)

$$
\{S_{BV}, S_{BV}\}_{BV} = \int_{T[1]M} \mu Tr \left[\frac{\delta_R S_{BV}}{\delta \underline{a}} \frac{\delta_L S_{BV}}{\delta \underline{a}} \right] .
$$

Using (2.33), we have

$$
\begin{aligned}
& \int_{T[1]M} \mu Tr [2k \left(F_{\underline{A_0}} + \underline{D}_{A_0}\underline{a} + \underline{aa} \right) 2k \left(F_{\underline{A_0}} + \underline{D}_{A_0}\underline{a} + \underline{aa} \right)] = \\
& = 4k^2 \int_{T[1]M} \mu Tr [\left(F_{\underline{A_0}} + \underline{D}_{A_0}\underline{a} + \underline{aa} \right) \left(F_{\underline{A_0}} + \underline{D}_{A_0}\underline{a} + \underline{aa} \right)] = \\
& = 4k^2 \int_{T[1]M} \mu Tr [\underline{F}_{\underline{A_0}}\underline{F}_{\underline{A_0}} + \underline{F}_{\underline{A_0}}\underline{D}_{\underline{A_0}}\underline{a} + \underline{F}_{\underline{A_0}}\underline{aa} + \underline{D}_{\underline{A_0}}\underline{a}\underline{F}_{\underline{A_0}} + \\
& + \underline{D}_{\underline{A_0}}\underline{a}\underline{D}_{\underline{A_0}}\underline{a} + \underline{D}_{\underline{A_0}}\underline{aaa} + \underline{aa}\underline{F}_{\underline{A_0}} + \underline{aa}\underline{D}_{\underline{A_0}}\underline{a} + \underline{aaaa}] = \\
& = 4k^2 \int_{T[1]M} \mu Tr [\underline{F}_{\underline{A_0}}\underline{F}_{\underline{A_0}} + 2\underline{F}_{\underline{A_0}}\underline{D}_{\underline{A_0}}\underline{a} + 2\underline{aa}\underline{F}_{\underline{A_0}} + \\
& + 2\underline{aa}\underline{D}_{\underline{A_0}}\underline{a} + \underline{D}_{\underline{A_0}}\underline{a}\underline{D}_{\underline{A_0}}\underline{a} + \underline{aaaa}] = \\
& = 4k^2 \int_{T[1]M} \mu Tr [\underline{F}_{\underline{A_0}}\underline{F}_{\underline{A_0}}] + 4k^2 \int_{T[1]M} \mu Tr [2\underline{F}_{\underline{A_0}}\underline{D}_{\underline{A_0}}\underline{a} + 2\underline{aa}\underline{F}_{\underline{A_0}} + \\
& + 2\underline{aa}\underline{D}_{\underline{A_0}}\underline{a} + \underline{D}_{\underline{A_0}}\underline{a}\underline{D}_{\underline{A_0}}\underline{a} + \underline{aaaa}] = \\
& = 4k^2 \int_{T[1]M} \mu Tr [\underline{F}_{\underline{A_0}}\underline{F}_{\underline{A_0}}] + 4k^2 \int_{T[1]M} \mu Tr [2\underline{F}_{\underline{A_0}}(\underline{D}_{\underline{A_0}}\underline{a} + \underline{aa}) + \\
& + (\underline{aa} + \underline{D}_{\underline{A_0}}\underline{a})(\underline{aa} + \underline{D}_{\underline{A_0}}\underline{a})].
\end{aligned}
$$

Now, we consider

$$
\begin{aligned}
& \underline{d} Tr \left[2\underline{a}\underline{F}_{\underline{A_0}} + \underline{a}\underline{D}_{\underline{A_0}} + \frac{2}{3}\underline{aaa} \right] = \\
& = Tr \left[\underline{D}_{\underline{A_0}} (2\underline{a}\underline{F}_{\underline{A_0}} + \underline{a}\underline{D}_{\underline{A_0}}\underline{a} + \frac{2}{3}\underline{aaa}) \right] = \\
& = Tr \Big[2\underline{D}_{\underline{A_0}}\underline{a}\underline{F}_{\underline{A_0}} - 2\underline{a}\underline{D}_{\underline{A_0}}\underline{F}_{\underline{A_0}} + \underline{D}_{\underline{A_0}}\underline{a}\underline{D}_{\underline{A_0}}\underline{a} + \\
& - \underline{D}_{\underline{A_0}}\underline{D}_{\underline{A_0}}\underline{a} + \frac{2}{3} \Big(\underline{D}_{\underline{A_0}}\underline{aaa} - \underline{a}\underline{D}_{\underline{A_0}}\underline{aa} + \underline{aa}\underline{D}_{\underline{A_0}}\underline{a} \Big) \Big] =
\end{aligned}
$$

$$= Tr[2\underline{F}_{\underline{A}_0}\underline{D}_{\underline{A}_0}\underline{a} + \underline{D}_{\underline{A}_0}\underline{a}\underline{D}_{\underline{A}_0}\underline{a} - \underline{a}(\underline{F}_{\underline{A}_0}\underline{a} - \underline{a}\underline{F}_{\underline{A}_0}) + 2\underline{a}\underline{a}\underline{D}_{\underline{A}_0}\underline{a} + \underline{a}\underline{a}\underline{a}\underline{a}] =$$
$$= Tr[2\underline{F}_{\underline{A}_0}\underline{D}_{\underline{A}_0}\underline{a} + \underline{D}_{\underline{A}_0}\underline{a}\underline{D}_{\underline{A}_0}\underline{a} + \underline{a}\underline{a}\underline{D}_{\underline{A}_0}\underline{a} + \underline{D}_{\underline{A}_0}\underline{a}\underline{a}\underline{a} + \underline{a}\underline{a}\underline{a}\underline{a} + 2\underline{F}_{\underline{A}_0}\underline{a}\underline{a}] =$$
$$= Tr[2\underline{F}_{\underline{A}_0}(\underline{D}_{\underline{A}_0}\underline{a} + \underline{a}\underline{a}) + (\underline{D}_{\underline{A}_0}\underline{a} + \underline{a}\underline{a})(\underline{D}_{\underline{A}_0}\underline{a} + \underline{a}\underline{a})].$$

Where we used the well-known relation

$$\underline{D}_{\underline{A}_0}\underline{F}_{\underline{A}_0} = 0 \tag{2.34}$$

called Bianchi identity.
We also used the Ricci identity

$$\underline{D}_{\underline{A}_0}\underline{D}_{\underline{A}_0}\underline{a} = \underline{F}_{\underline{A}_0}\underline{a} - \underline{a}\underline{F}_{\underline{A}_0}, \tag{2.35}$$

and the following algebraic relation

$$Tr(\underline{a}\underline{a}\underline{a}\underline{a}) = 0.$$

It follows that:

$$\begin{aligned} &\int_{T[1]M} \mu Tr[2\underline{F}_{\underline{A}_0}(\underline{D}_{\underline{A}_0}\underline{a} + \underline{a}\underline{a}) + (\underline{D}_{\underline{A}_0}\underline{a} + \underline{a}\underline{a})(\underline{D}_{\underline{A}_0}\underline{a} + \underline{a}\underline{a})] = \\ &= \int_{T[1]M} \mu \underline{d} Tr\left[2\underline{a}\underline{F}_{\underline{A}_0} + \underline{a}\underline{D}_{\underline{A}_0}\underline{a} + \frac{2}{3}\underline{a}\underline{a}\underline{a}\right] = 0 \end{aligned} \tag{2.36}$$

By Stokes' theorem.
Next, one has

$$\int_{T[1]M} \mu Tr\left[\underline{F}_{\underline{A}_0}\underline{F}_{\underline{A}_0}\right] = 0. \tag{2.37}$$

Since the integrand of equation (2.37) is of form degree 4.
Using (2.36) and (2.37), one gets (2.31). □

2.3.4 Quantum BV equation

We now discuss quantum BV Chern-Simons theory.
From the theory of the BV formalism we know that action (2.27) satisfies the Quantum BV master equation, namely

$$\{S_{BV}, S_{BV}\} - 2i\hbar\Delta_{BV}S_{BV} = 0 \tag{2.38}$$

Since S_{BV} obeys to the classical BV master equation, we can rewrite (2.38) as

$$2i\hbar\Delta_{BV}S_{BV} = \Delta_{BV}S_{BV} = 0, \tag{2.39}$$

where Δ_{BV} is the BV laplacian. It has the following form

$$\Delta_{BV} = \frac{1}{2}\int_{T[1]M} \mu Tr\left(\frac{\delta_L^2}{\delta\underline{A}\delta\underline{A}}\right) \tag{2.40}$$

Proof of relation (2.39)
Using the BV action (2.26) we have

$$\Delta_{BV} S_{BV} = \Delta_{BV} k \int_{T[1]M} \mu Tr \left[\underline{A} d \underline{A} + \frac{2}{3} \underline{A}\underline{A}\underline{A} \right]$$

Recalling that

$$\frac{\delta_L S_{BV}}{\delta \underline{A}} = \frac{\delta_R S_{BV}}{\delta \underline{A}} = 2 \underline{F}_{\underline{A}} \tag{2.41}$$

then, we have

$$\begin{aligned} k \int_{T[1]M} \mu Tr \left(\frac{\delta_L}{\delta \underline{A}} \underline{F}_{\underline{A}} \right) &= \\ = k \int_{T[1]M} \mu Tr [D_{\underline{A}} (\delta(0) 1_{\mathfrak{g}})] &= \\ = dim \mathfrak{g} \; \delta(0) \; k \int_{T[1]M} \mu Tr [D_{\underline{A}} 1] &= 0 \end{aligned}$$

Where $\delta(0)$ is the Dirac delta function evaluated in 0 and $dim\mathfrak{g}$ is the dimension of the Lie algebra $\mathfrak{g}$. We impose a cut-off in order to regularize the divergence that comes from the Dirac delta function.
In a suitable regularization scheme, this result is supposed to be valid before removing the regularization, in spite of the fact that $\delta(0)$ tends to infinity. □

2.3.5 Nilpotence and Invariance

We introduce the BV field variations as follows

$$\delta_{BV} \underline{a} = \frac{1}{2k} (S_{BV}, \underline{a}) = \underline{F}_{\underline{A_0}} + \underline{D}_{\underline{A_0}} \underline{a} + \underline{a}\underline{a}. \tag{2.42}$$

We can write the action (2.27) in field components using the decomposition (2.13). One has:

$$\begin{aligned} S_{BV} = k S_{CS}(A_0) + k \int_M \mu Tr \Big[& 2a F_{A_0} + a D_{A_0} a + \\ & + \frac{2}{3} aaa - 2a^\dagger (D_{A_0} c + [a, c]) - 2c^\dagger cc \Big], \end{aligned} \tag{2.43}$$

and we have

$$\delta_{BV} c = cc \tag{2.44}$$

$$\delta_{BV} a = D_{A_0} c + [a, c] \tag{2.45}$$

$$\delta_{BV} a^\dagger = F_{A_0} + D_{A_0} a + aa - \left[a^\dagger, c \right] \tag{2.46}$$

$$\delta_{BV} c^\dagger = -D_{A_0} a^\dagger - \left[c, c^\dagger \right] - \left[a, a^\dagger \right] \tag{2.47}$$

The BV field variations (2.42) enjoys the nilpotence property, i.e.

$$\delta_{BV}^2 \underline{a} = 0 \tag{2.48}$$

Proof of relation (2.48). Using equation (2.42), we have

$$\begin{aligned}\delta_{BV}^2\underline{a} &= \delta_{BV}\left(\frac{1}{2k}\left\{S_{BV},\underline{a}\right\}_{BV}\right) = \\ &= \delta_{BV}(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa}) = \\ &= -\underline{D_{A_0}}\delta_{BV}\underline{a} + \delta_{BV}\underline{aa} - \underline{a}\delta_{BV}\underline{a}.\end{aligned}$$

Using again equation (2.42), we have

$$\begin{aligned}= -\underline{D_{A_0}}(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa}) + (\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa})\underline{a} - \underline{a}(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa}) = \\ = -\underline{D_{A_0}}\underline{F_{A_0}} - \underline{D^2_{A_0}a} - \underline{D_{A_0}aa} + \underline{aD_{A_0}a} + \underline{F_{A_0}a} + \\ + \underline{D_{A_0}aa} + +\underline{aaa} - \underline{aF_{A_0}} - \underline{aD_{A_0}a} - \underline{aaa} = \\ = -\underline{F_{A_0}a} + \underline{aF_{A_0}} + \underline{F_{A_0}a} - \underline{aF_{A_0}} = 0.\end{aligned}$$

The action (2.27) is invariant under the BV field variations (2.42), namely

$$\delta_{BV}S_{BV} = 0 \tag{2.49}$$

Proof of relation (2.49):

$$\delta_{BV}S_{BV} = \delta_{BV}\left\{kS_{CS}(A_0) + k\int_{T[1]M}\mu Tr\left[2\underline{aF_{A_0}} + \underline{aD_{A_0}a} + \frac{2}{3}\underline{aaa}\right]\right\}.$$

Since $\delta_{BV}S_{CS}(A_0) = 0$, we obtain

$$k\int_{T[1]M}\mu Tr\Big[2\delta_{BV}\underline{aF_{A_0}} + \delta_{BV}\underline{aD_{A_0}a} + \underline{aD_{A_0}}\delta_{BV}\underline{a} + \frac{2}{3}(\delta_{BV}\underline{aaa} - \underline{a}\delta_{BV}\underline{aa} + \underline{aa}\delta_{BV}\underline{a})\Big].$$

Using the relation (2.42)

$$\begin{aligned}&k\int_{T[1]M}\mu Tr\Big[2(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa})\underline{F_{A_0}} + (\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa})\underline{D_{A_0}a} + \underline{aD_{A_0}}(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa}) + \\ &+ \frac{2}{3}\Big\{(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa})\underline{aa} + \underline{a}(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa})\underline{a} + \underline{aa}(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa})\Big\}\Big] = \\ &= k\int_{T[1]M}\mu Tr[2(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa})\underline{F_{A_0}} + (\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa})\underline{D_{A_0}a} + \\ &+ \underline{aD_{A_0}}(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa}) + 2\underline{aa}(\underline{F_{A_0}} + \underline{D_{A_0}a} + \underline{aa}) = \\ &k\int_{T[1]M}\mu Tr[2\underline{F_{A_0}F_{A_0}} + 2\underline{D_{A_0}F_{A_0}} + 2\underline{aaF_{A_0}} + \underline{F_{A_0}}\underline{D}_{A_0}\underline{a} + \underline{D_{A_0}aD_{A_0}a} + \underline{aaD_{A_0}a} + \\ &\qquad + \underline{aD_{A_0}F_{A_0}} + \underline{aD_{A_0}D_{A_0}a} + \underline{aD_{A_0}aa} + 2\underline{aaF_{A_0}} + 2\underline{aaD_{A_0}a} + 2\underline{aaaa}] = \\ &= k\int_{T[1]M}\mu Tr[2\underline{F_{A_0}F_{A_0}} + 4\underline{aaF_{A_0}} + 4\underline{F_{A_0}D_{A_0}a} + \underline{D_{A_0}aD_{A_0}a} + 4\underline{aaD_{A_0}} + \underline{aD_{A_0}D_{A_0}a} + 2\underline{aaa} = \\ &\quad = k\int_{T[1]M}\mu\Big(-dTr\left\{\underline{F_{A_0}a} + \underline{aD_{A_0}a} + \underline{aaa}\right\} + 2Tr[\underline{F_{A_0}F_{A_0}} + \\ &\quad + 2\underline{F_{A_0}}(\underline{D_{A_0}a} + \underline{aa}) + (\underline{D_{A_0}a} + \underline{aa})(\underline{D_{A_0}a} + \underline{aa})]\Big) = \\ &\quad = 2k\int_{T[1]M}\mu Tr[\underline{F_{A_0}F_{A_0}} + 2\underline{F_{A_0}}(\underline{D_{A_0}a} + \underline{aa}) + (\underline{D_{A_0}a} + \underline{aa})(\underline{D_{A_0}a} + \underline{aa})]\end{aligned}$$

The last expression was shown to vanish earlier, indeed one gets (2.49). □

(2.49) corresponds to the classical master equation (2.31).

2.3.6 Gauge fixing of the BV Chern-Simons theory

In this section we discuss the gauge fixing procedure, that is necessary to quantize the Chern-Simons theory. First of all we introduce trivial pairs of fields and antifields

$$\widetilde{c} \in \Gamma(M, AdP[+1]) \qquad \widetilde{c}^{\dagger} \in \Gamma(M, \Lambda^3 T^* M \otimes AdP[0]) \tag{2.50}$$

$$\widetilde{\gamma} \in \Gamma(M, AdP[0]) \qquad \widetilde{\gamma}^{\dagger} \in \Gamma(M, \Lambda^3 T^* M \otimes AdP[-1]) \tag{2.51}$$

The auxiliary BV symplectic form is

$$\Omega_{BVaux} = \int_M \mu Tr \left[\delta \widetilde{c}^{\dagger} \delta \widetilde{c} + \delta \widetilde{\gamma}^{\dagger} \delta \widetilde{\gamma} \right] \tag{2.52}$$

The corresponding auxiliary BV bracket are

$$\{F, G\}_{BVaux} = \int_M \mu Tr \left[\frac{\delta_R F}{\delta \widetilde{c}} \frac{\delta_L G}{\delta \widetilde{c}^{\dagger}} - \frac{\delta_R F}{\delta \widetilde{c}^{\dagger}} \frac{\delta_L G}{\delta \widetilde{c}} + \frac{\delta_R F}{\delta \widetilde{\gamma}} \frac{\delta_L G}{\delta \widetilde{\gamma}^{\dagger}} - \frac{\delta_R F}{\delta \widetilde{\gamma}^{\dagger}} \frac{\delta_L G}{\delta \widetilde{\gamma}} \right] \tag{2.53}$$

Auxiliary BV action

We can introduce the auxiliary BV action as follows

$$S_{BVaux} = -2 \int_M Tr \left(\widetilde{c}^{\dagger} \widetilde{\gamma} \right) \tag{2.54}$$

From a direct inspection the following relation holds true

$$\{S_{BVaux}, S_{BVaux}\}_{BVaux} = 0, \tag{2.55}$$

that is the BV classical master equation for auxiliary action (2.54).
We introduce the auxiliary BV variation as follows

$$\delta_{BVaux} = \frac{1}{2} \left(S_{BVaux}, \cdot \right)_{BVaux} . \tag{2.56}$$

We apply (2.56) to fields and antifields introduced in equations (2.50) and (2.51), then

$$\delta_{BVaux} \widetilde{c} = -\widetilde{\gamma} \tag{2.57}$$

$$\delta_{BVaux} \widetilde{\gamma} = 0 \tag{2.58}$$

$$\delta_{BVaux} \widetilde{c}^{\dagger} = 0 \tag{2.59}$$

$$\delta_{BVaux} \widetilde{\gamma}^{\dagger} = -\widetilde{c}^{\dagger} \tag{2.60}$$

As expected (2.56) enjoys the nilpotence property, namely

$$\delta^2_{BVaux}(\cdot) = 0 \tag{2.61}$$

Proof of relation (2.61)

$$\delta^2_{BVaux}(\widetilde{c}, \widetilde{c}^\dagger, \widetilde{\gamma}, \widetilde{\gamma}^\dagger) = \delta_{BVaux}(-\widetilde{\gamma}, 0, 0, -\widetilde{c}^\dagger) = 0$$

thanks to (2.58) and (2.59), (2.61) holds true. □

The auxiliary BV action (2.54) is invariant under auxiliary BV variations (2.56), namely

$$\delta_{BVaux} S_{BVaux} = 0, \tag{2.62}$$

which corresponds to the BV classical master equation (2.55) for the auxiliary BV action (2.54).

Proof of relation (2.62)

$$\begin{aligned}\delta_{BVaux} S_{BVaux} &= \delta_{BVaux}\left(-2k\int_M Tr\left(\widetilde{c}^\dagger\widetilde{\gamma}\right)\right) = \\ &= -2k\int_M Tr\left(\delta_{BVaux}\widetilde{c}^\dagger\widetilde{\gamma} + \widetilde{c}^\dagger\delta_{BVaux}\widetilde{\gamma}\right)\end{aligned}$$

Using the relation (2.58) and (2.59), one gets equation (2.62). □

2.3.7 The gauge fermion

The gauge fermion for the BV 3d Chern-Simons theory is

$$\Psi = \int_M Tr\left(\widetilde{c} D_{A_0} \star a\right), \tag{2.63}$$

where $\star$, as usual, is the Hodge operator.
Using (2.63) we can define a lagrangian submanifold $\mathscr{L}$ in the field space as follows

$$\varphi^\dagger_A = \frac{\delta_l \Psi}{\delta \varphi^A}. \tag{2.64}$$

One finds

$$c^\dagger = \frac{\delta_l \Psi}{\delta c} = 0 \tag{2.65}$$

$$a^\dagger = \frac{\delta_l \Psi}{\delta a} = \star D_{A_0}\widetilde{c} \tag{2.66}$$

$$\widetilde{c}^\dagger = \frac{\delta_l \Psi}{\delta \widetilde{c}} = D_{A_0} \star a \tag{2.67}$$

$$\widetilde{\gamma}^\dagger = \frac{\delta_l \Psi}{\delta \widetilde{\gamma}} = 0 \tag{2.68}$$

Thanks to the gauge fermion (2.63) we can define the gauge fixed action as follows:

$$\begin{aligned}I &= (S_{BV} + S_{BVaux})|_{\mathscr{L}} = \\ &= (kS_{CS}(A_0) + kS_{CS}(\underline{a}) + S_{BVaux})\,|_{\mathscr{L}} =\end{aligned} \tag{2.69}$$

$$= \left\{ kS_{CS}(A_0) + k\int_M Tr\left[2aF_{A_0} + aD_{A_0}a + \frac{2}{3}aaa\right] - 2k\int_M Tr\left(\widetilde{c}^{\dagger}\widetilde{\gamma}\right)\right\}\Bigg|_{\mathscr{L}} =$$

Using (2.64)

$$\begin{aligned} = kS_{CS}(A_0) + k\int_M Tr\Big[2aF_{A_0} + aD_{A_0}a + \frac{2}{3}aaa + \\ - 2\widetilde{\gamma}D_{A_0}\star a + 2D_{A_0}\widetilde{c}\star(D_{A_0}c + [a,c])\Big] \end{aligned} \tag{2.70}$$

We can introduce now the BRST operator

$$s = \delta_{BV}|_{fields}, \tag{2.71}$$

such that:

$$sc = \delta_{BV}c = cc \tag{2.72}$$

$$sa = \delta_{BV}a = D_{A_0}c + [a,c] \tag{2.73}$$

$$s\widetilde{c} = \delta_{BVaux}\widetilde{c} = -\widetilde{\gamma} \tag{2.74}$$

$$s\widetilde{\gamma} = \delta_{BVaux}\widetilde{\gamma} = 0 \tag{2.75}$$

Operator (2.71) is nilpotent, i.e

$$s^2 = 0. \tag{2.76}$$

This is a trivial relation because the BRST formalism works for this model.

Proof of relation (2.76).
Consider s^2 acting on the fields $a, c, \widetilde{c}, \widetilde{\gamma}$, then

$$\delta_{BV}(\delta_{BV}a + \delta_{BV}c + \delta_{BV}\widetilde{c} + \delta_{BV}\widetilde{\gamma})$$

Using relations (2.72),(2.73),(2.74) and (2.75), we obtain

$$\delta_{BV}(D_{A_0}c + [a,c] + cc - \widetilde{\gamma})$$

Using again the previous relations, we have

$$\begin{aligned} &- D_{A_0}(\delta_{BV}c) + [\delta_{BV}a, c] + [a, \delta_{BV}c] + \delta_{BV}cc - c\delta_{BV}c = \\ &= -D_{A_0}(cc) + (D_{A_0}c)c + [a,c]c - c(D_{A_0}c) + c[a,c] + \\ &- acc + cca + ccc - ccc = \\ &= -D_{A_0}(cc) + (D_{A_0}c) - cac - c(D_{A_0}c) + cac = 0 \end{aligned}$$

relation (2.76) holds true. □

The gauge fixed action (2.69) is invariant under the BRST operator, namely

$$sI = 0 \tag{2.77}$$

Proof of relation (2.77)

$$\begin{aligned}
sI &= s(S_{BV} + S_{BVaux})|_{\mathscr{L}} = \\
&= s\ kS_{CS}(A_0) + s\ k\int_M Tr\Big[2aF_{A_0} + aD_{A_0}a + \\
&+ \frac{2}{3}aaa + 2D_{A_0}\widetilde{c} \star (D_{A_0}c + [a,c]) - 2\gamma D_{A_0} \star a\Big]
\end{aligned}$$

Since $sS_{CS}(A_0) = \delta_{BV}S_{CS}(A_0) = 0$, we have

$$\begin{aligned}
&= s\ k\int_M Tr\Big[2aF_{A_0} + aD_{A_0}a + \frac{2}{3}aaa + \\
&+ 2D_{A_0}\widetilde{c} \star (D_{A_0}c + [a,c]) - 2\gamma D_{A_0} \star a\Big] = \\
&= k\int_M Tr[2saF_{A_0} + saD_{A_0}a + aD_{A_0}sa + 2saaa + \\
&- 2D_{A_0}s\widetilde{c} \star (D_{A_0} + [a,c]) - 2D_{A_0}\widetilde{c} \star (-D_{A_0}sc + \\
&+ [sa,c] - [a,sc]) - 2s\gamma D_{A_0} \star a - 2\gamma D_{A_0} \star sa] = \\
&= k\int_M Tr[2(D_{A_0}c + [a,c])F_{A_0} + (D_{A_0}c + [a,c])D_{A_0}a + \\
&+ aD_{A_0}(D_{A_0}c + [a,c]) + 2aa(D_{A_0}c + [a,c]) + \\
&+ 2D_{A_0}\gamma \star (D_{A_0}c + [a,c]) - 2D_{A_0}\widetilde{c} \star (D_{A_0}(cc) + \\
&- [D_{A_0}c + [a,c], c] - [a,cc]) + 2\gamma D_{A_0} \star ((D_{A_0}c + [a,c]))] =
\end{aligned}$$

$$\begin{aligned}
&= k\int_M Tr[2D_{A_0}\gamma \star (D_{A_0}c + [a,c]) - 2D_{A_0}\gamma \star (D_{A_0}c + [a,c]) + \\
&- 2D_{A_0}\widetilde{c} \star (D_{A_0}cc - cD_{A_0}c - D_{A_0}cc + cD_{A_0}c - (ac + ca)c + \\
&+ c(ac + ca) + acc - cca) + 2F_{A_0}(D_{A_0}c + [a,c]) + \\
&+ D_{A_0}a(D_{A_0}c + [a,c]) + D_{A_0}a(D_{A_0}c + [a,c]) + 2aa(D_{A_0}c + [a,c])] = \\
&= 2k\int_M Tr[F_{A_0}(D_{A_0}c + [a,c]) + (D_{A_0}a + aa)(D_{A_0}c + [a,c])]
\end{aligned}$$

Now, one has

$$0 = \int_M dTr\,[(F_{A_0} + D_{A_0}a + aa)c] =$$

$$\begin{aligned}
&= \int_M Tr\,\{D_{A_0}[(F_{A_0} + D_{A_0}a + aa)c] + [a, (F_{A_0} + D_{A_0}a + aa)c]\} = \\
&= \int_M Tr[D_{A_0}F_{A_0}c + F_{A_0}D_{A_0}c + D_{A_0}D_{A_0}ac + D_{A_0}aD_{A_0}c + D_{A_0}aac + \\
&- aD_{A_0}ac + aaD_{A_0}c + [a,F_{A_0}]c + F_{A_0}[a,c] + [a,D_{A_0}a]c + D_{A_0}a[a,c] + \\
&+ [a,aa]c + aa[a,c]] = \\
&= \int_M Tr[F_{A_0}D_{A_0}c + [F_{A_0},a]c + D_{A_0}aD_{A_0}c - [a,D_{A_0}a]c + aaD_{A_0}c + \\
&- [F_{A_0},a]c + F_{A_0}[a,c] + [a,D_{A_0}a]c + D_{A_0}a[a,c] + aa[a,c]] = \\
&= \int_M Tr[F_{A_0}(D_{A_0}c + [a,c]) + (D_{A_0}a + aa)(D_{A_0}c + [a,c])]
\end{aligned}$$

From the previous relation, (2.77) follows.

Chapter 3

BF theory

3.1 Ordinary BF theory

3.1.1 Introduction

In this chapter we provide an introduction to the so called BF theory, probably introduced by Horowitz in [18]. This is an important topological quantum field theory of Schwarz type because its deformations have a relevant role in studies about the gravity quantization.

3.1.2 Classical action and symmetries

In this section we introduce BF theory, begging with some elements of its geometry.

Consider a G-bundle $P \longrightarrow M$, where M is a connected, orientable and closed manifold of dimension $m \geq 2$. G is a compact simple Lie group with a Lie algebra $\mathfrak{g}$. We assume the bundle to be trivial, $P = M \times G$. We denote by $\Omega(M)$ the space of differential forms on M and by $\Omega(M, \mathrm{Ad}P)$ the space of differential forms on M with values in the adjoint bundle $\mathrm{Ad}P = P \times_{\mathrm{Ad}} \mathfrak{g}$ (which are $\mathfrak{g}$-valued forms on M). The field content of the BF theory is the following

(i) A connection $A \in \Omega^1(M, \mathrm{Ad}P)$ which plays the role of dynamical field. Its curvature denoted by F_A is a form in $\Omega^2(M, \mathrm{Ad}P)$.

(ii) A form $B \in \Omega^{m-2}(M, \mathrm{Ad}P)$.

Using these fields, we can construct the action of the model

$$S_{BF} = k \int_M \langle B, F_A \rangle \,, \tag{3.1}$$

where $k \in \mathbb{R}$ is a constant and $\langle \cdot, \cdot \rangle$ is an invariant, non singular, bilinear form on $\mathfrak{g}$. If we consider this form to be realized by the trace over some representation of the Lie algebra $\mathfrak{g}$, we can rewrite (3.1) as follows

$$S_{BF} = k \int_M Tr(BF_A). \tag{3.2}$$

Let's calculate now the equations of motion. From the Euler-Lagrange equation, one finds:

$$\frac{\delta S_{BF}}{\delta B} = kF_A = 0 \tag{3.3}$$

$$\frac{\delta S_{BF}}{\delta A} = kD_A B = 0. \tag{3.4}$$

Eq. (3.3) corresponds to the condition of flatness for the connection A.
Next we discuss the symmetries of the BF theory.
Action (3.2) is invariant under the finite gauge transformations

$$A \longrightarrow {}^gA = gAg^{-1} + dgg^{-1} \tag{3.5}$$

$$B \longrightarrow {}^gB = gBg^{-1}, \tag{3.6}$$

where $g \in Map(M, G)$, the group of continuous maps $M \longrightarrow G$.
We can provide an infinitesimal form for the previous gauge transformations, namely

$$\delta A = -D_A\epsilon \tag{3.7}$$

$$\delta B = -([B, \epsilon] + D_A\tau), \tag{3.8}$$

where $\epsilon \in \Omega^0(M, AdP)$ and τ is an element of $\Omega^{m-3}(M, AdP)$.

3.1.3 Deformations of the BF theory

We discuss now all the possible deformations of the BF theory. To obtain them is necessary add to the action (3.2) a topological term which is invariant under gauge transformations (3.5) and (3.6). We don't consider Chern-Simons terms, which exist only in odd dimensions and are invariant under transformation (3.7). Imposing that B be a $(m-2)$-form which takes value in the adjoint bundle, we have the following theorem:

Theorem 3.1 Exists topological, gauge invariant and non-singular deformations of the BF theory only for $m = 2, 3, 4$. They have the following form

$$S^{(2)}_{BF} = \int_M Tr(f(B)F_A) \qquad m = 2 \tag{3.9}$$

$$S^{(3)}_{BF} = \int_M Tr\left(B \wedge F_A + \frac{\Lambda}{3!} B \wedge B \wedge B\right) \qquad m = 3 \tag{3.10}$$

$$S^{(4)}_{BF} = \int_M Tr\left(B \wedge F_A + \frac{\Lambda}{2} B \wedge B\right) \qquad m = 4 \tag{3.11}$$

where $k \in \mathbb{R}$ and $f(B)$ is an analytic function on the space $\Omega^0(M, AdP)$.

Proof of the theorem 3.1
Consider the general form for a deformed action defined on a $m-$dimensional manifold M, which is invariant under transformations (3.5) and (3.6), namely

$$S^{(m)}_{BF} = \int_M \sum_{r\geq 0, s\geq 0} D^r_A B^s \tag{3.12}$$

In order to obtain a topological action we require

$$r + (m-2)s = m \tag{3.13}$$

with r, s positive integers.
To solve (3.13) we can consider many cases, namely

(I) For **m=2**, (3.13) has trivially one solution for $s = 0$ and $r = 2$. Since $D_A^2 a = aF_A$, we have action (3.9) with $f(B) = \sum_{s=0}^{\infty} \alpha_s B^s$,where α_s are arbitrary constants.

(II) For **m**$\geq$ **2** we can rewrite (3.13) as

$$s = \frac{m-r}{m-2} \tag{3.14}$$

it has solution if $r = 0, 1, 2, m$. In what follows each solution is considered singularly.

(a) **r=m**. In this case $s = 0$. Since $D_A^2 a = aF_A$, the we obtain $F^{\frac{m}{2}}$ if m is even, while, if m is odd, we have $D_A F^{\frac{m-1}{2}}$ which vanishes thanks to the Bianchi identity.

(b) **r=2**. In this case $s = 1$, then we have the ordinary BF term.

(c) **r=1**. In this case $s = \frac{m-1}{m-2}$ which has a solution only for $m = 3$. We obtain the term $BD_A B$.

(d) **r=0**. In this case $s = \frac{m}{m-2}$. It has a solution only for $m = 3$ and $m = 4$. We obtain $s = 3$ and $s = 2$ respectively.
We have terms $B \wedge B \wedge B$ and $B \wedge B$ which appear in (3.10) and (3.11).

3.2 BV BF theory

3.2.1 Introduction

In this section we discuss the BF theory in a BV perspective. First of all we introduce the geometrical framework for a BV BF theory in a arbitrary dimension m, then we define the action of the model and we demonstrate the BV master equation. Next we will take a specific dimension and we face the problem of the gauge fixing procedure..

3.2.2 Geometrical framework

In this model the geometrical framework is constituted by the following components:

(1) An oriented, smooth, compact m-fold M

(2) A principal G bundle P over M. G is a compact Lie group

In this model we adopt the superfield formalism. The base in this example is the degree $+1$ shifted tangent bundle of M, $T[1]M$.
In thid case the related bundle projection is

$$\Pi : T[1]M \longrightarrow M. \tag{3.15}$$

We consider the adjoint bundle of P, $\mathrm{Ad}P$, which is a vector bundle over M, while $\Pi\mathrm{Ad}P$ is a vector bundle over $T[1]M$.
The superfield content of this model consists in the following fields

$$\underline{A} - \underline{A_0} \in \underline{\Gamma}(T[1]M, \mathrm{Ad}P) \tag{3.16}$$

$$\underline{B} \in \underline{\Gamma}(T[1]M, \mathrm{Ad}P), \tag{3.17}$$

which are respectively a 1-form and a $(m-2)$-form of adjoint type. $\underline{A_0}$ is an ordinary background connection of the bundle P. We denote with $\underline{\Gamma}$ the internal sections of the bundle P.
We can decompose (3.16) and (3.17) in field components as follows:

$$\underline{A} - \underline{A_0} = -c + a + (-1)^m b^\dagger + \sum_{k=1}^{m-2} (-1)^m \tau_k^\dagger \tag{3.18}$$

$$\underline{B} = \sum_{k=1}^{m-2} \tau_k + b + a^\dagger - c^\dagger \tag{3.19}$$

Each component in (3.18) and (3.19) has a definite form-ghost bidegree, namely

$\underline{A}$		$\underline{B}$	
c	$(0,1)$	τ_k	$(0,m-2)$
a	$(1,0)$	τ_{k-1}	$(1,m-3)$
$b^\dagger$	$(2,-1)$	τ_{k-2}	$(2,m-4)$
$\tau_1^\dagger$	$(3,-2)$	...	...
$\tau_2^\dagger$	$(4,-3)$	τ_1	$(m-3,1)$
$\tau_3^\dagger$	$(5,-4)$	b	$(m-2,0)$
...	...	$a^\dagger$	$(m-1,-1)$
$\tau_k^\dagger$	$(m,-m+1)$	$c^\dagger$	$(m,-2)$

We briefly discuss now the integration of superfields. We can perform this integration using the standard supermeasure μ of $T[1]M$, which as $T[-1]M$ degree $-m$. Given φ a superfield, one has the following relation

$$\int_{T[1]M} \mu\varphi = \int_M \varphi^{(m)}, \tag{3.20}$$

where $\varphi^{(m)}$ is the component of φ of $T[1]M$ which has standard form degree m.

BV symplectic form

In this theory is relevant the following symplectic form

$$\Omega_{BV} = \int_{T[1]M} \mu Tr(\delta\underline{B}\delta\underline{A}), \tag{3.21}$$

which has degree -1 and is closed as required, namely

$$\delta\Omega_{BV} = 0. \tag{3.22}$$

The related BV bracket reads as follows

$$\{F,G\}_{BV} = \int_{T[1]M} \mu Tr\left(\frac{\delta_R F}{\delta\underline{A}}\frac{\delta_L G}{\delta\underline{B}} - (-1)^m \frac{\delta_R F}{\delta\underline{B}}\frac{\delta_L G}{\delta\underline{A}}\right) \tag{3.23}$$

We can formulate (3.21) in field components. Using (3.18) and (3.19), we get

$$\Omega_{BV} = \int_M Tr\left[\delta a^\dagger\delta a + \delta b^\dagger\delta b + \delta c^\dagger\delta c + \sum_{k=1}^{m-2}\delta\tau_k^\dagger\delta\tau_k\right]. \tag{3.24}$$

we can rewrite (3.23) in field components as follows

$$\begin{aligned}\{F,G\}_{BV} = \int_M Tr\Bigg[&\frac{\delta_R F}{\delta a}\frac{\delta_L G}{\delta a^\dagger} - \frac{\delta_R F}{\delta a^\dagger}\frac{\delta_L G}{\delta a} + \frac{\delta_R F}{\delta b}\frac{\delta_L G}{\delta b^\dagger} - \frac{\delta_R F}{\delta b^\dagger}\frac{\delta_L G}{\delta b} + \\ &+ \frac{\delta_R F}{\delta c}\frac{\delta_L G}{\delta c^\dagger} - \frac{\delta_R F}{\delta c^\dagger}\frac{\delta_L G}{\delta c} + \sum_{k=1}^{m-2}\frac{\delta_R F}{\delta\tau_k}\frac{\delta_L G}{\delta\tau_k^\dagger} - \frac{\delta_R F}{\delta\tau_k^\dagger}\frac{\delta_L G}{\delta\tau_k}\Bigg]\end{aligned} \tag{3.25}$$

3.2.3 BV BF action

The action for the BF theory in a BV perspective is formally

$$S_{BV} = k \int_{T[1]M} \mu Tr(\underline{B}\underline{F}_{\underline{A}}), \tag{3.26}$$

where the curvature $\underline{F}_{\underline{A}}$ has the well-known form:

$$\underline{F}_{\underline{A}} = d\underline{A} + \frac{1}{2}[\underline{A}, \underline{A}] \tag{3.27}$$

BV Classical master equation

We can demonstrate the classical BV equation, namely

$$\{S_{BV}, S_{BV}\}_{BV} = 0 \tag{3.28}$$

Proof of equation (3.28)
In order to demonstrate the previous relation we have to calculate directional derivatives for both the superfields. Consider first the superfield $\underline{A}$, then we determine the following relation

$$\frac{d}{dt} S_{BV}(\underline{A} + t\underline{a}), \tag{3.29}$$

where $\underline{a} \in \Omega^1(M, \mathrm{Ad}P)$.
Using (3.26) in (3.29), we obtain

$$\frac{d}{dt} k \int_{T[1]M} \mu Tr[\underline{B}d\underline{A} + (-1)^{m-1}td\underline{B}\underline{a} + \underline{B}\underline{A}\underline{A} + t((-1)^{m-1}\underline{A}\underline{B}\underline{a} + \underline{B}\underline{A}\underline{a}) + t^2\underline{B}\underline{a}\underline{a}]\Bigg|_{t=0} =$$

$$= \frac{d}{dt} k \int_{T[1]M} \mu Tr[\underline{B}d\underline{A} + \underline{B}\underline{A}\underline{A} + (-1)^{m-1}t(d\underline{B} + [\underline{A}, \underline{B}])\underline{a} + t^2\underline{B}\underline{a}\underline{a}]\Bigg|_{t=0} =$$

$$= (-1)^{m-1} k \int_{T[1]M} \mu Tr[(d\underline{B} + [\underline{A}, \underline{B}])\underline{a}]$$

Hence

$$\frac{\delta_R S_{BV}}{\delta \underline{A}} = (-1)^{m-1} \frac{\delta_L S_{BV}}{\delta \underline{A}} = (-1)^{m-1} D_{\underline{A}}\underline{B} \tag{3.30}$$

For the superfield $\underline{B}$, we have to calculate

$$\frac{d}{dt} S_{BV}(\underline{B} + t\underline{b})\Bigg|_{t=0}. \tag{3.31}$$

Using (3.26) in (3.31), we have

$$\frac{d}{dt} k \int_{T[1]M} \mu Tr[(\underline{B} + t\underline{b})\underline{F}_{\underline{A}}] = \int_{T[1]M} \mu Tr[\underline{b}\underline{F}_{\underline{A}}]$$

Hence

$$\frac{\delta_L S_{BV}}{\delta \underline{B}} = \frac{\delta_R S_{BV}}{\delta \underline{B}} = \underline{F}_{\underline{A}} \tag{3.32}$$

Substituting (3.30) and (3.32) in (3.28), we have

$$
\begin{aligned}
\{S_{BV}, S_{BV}\} = 2\int_{T[1]M} \mu Tr\left[\frac{\delta_R S_{BV}}{\delta \underline{A}}\frac{\delta_L S_{BV}}{\delta \underline{B}}\right] = 2k^2 \int_{T[1]M} \mu Tr[\underline{F}_{\underline{A}} D_{\underline{A}} \underline{B}] = \\
= 2k^2 \int_{T[1]M} \mu Tr[D_{\underline{A}}(\underline{F}_{\underline{A}}\underline{B})] - 2k^2 \int_{T[1]M} \mu Tr[D_{\underline{A}}\underline{F}_{\underline{A}}\underline{B}] = 0
\end{aligned}
$$

where the second term vanishes thanks to the Bianchi identity

$$
D_{\underline{A}}\underline{F}_{\underline{A}} = 0, \tag{3.33}
$$

while the first term vanishes thanks to the Stokes' theorem, therefore relation (3.28) holds true. □

3.2.4 Nilpotence and Invariance

In this section we introduce the BV field variations and their properties.
We define the BV field variations as follows

$$
\delta_{BV}(\cdot) = \frac{1}{k}\{S_{BV}, \cdot\}_{BV}, \tag{3.34}
$$

where $\{\cdot,\cdot\}_{BV}$ is the BV bracket introduced in (3.23).
For the superfields $\underline{A}$ and $\underline{B}$ we have the following relation

$$
\delta_{BV}\underline{A} = \frac{1}{k}\{S_{BV}, \underline{A}\}_{BV} = \underline{F}_{\underline{A}} \tag{3.35}
$$

$$
\delta_{BV}\underline{B} = \frac{1}{k}\{S_{BV}, \underline{B}\}_{BV} = -D_{\underline{A}}\underline{B} \tag{3.36}
$$

Proof of relation (3.35) Considering the following test function

$$
\langle \underline{A}, \varphi\rangle = \int_{T[1]M} \mu Tr(\underline{A}\varphi) \tag{3.37}
$$

Substituting (3.37) in (3.34) we have

$$
\begin{aligned}
& -\delta\underline{A} = -\{S_{BV}, \langle\underline{A},\varphi\rangle\}_{BV} = -(-1)^{-m}\langle\{S_{BV}, \underline{A}\}_{BV}, \varphi\rangle = \\
& = -(-1)^{-m}\int_{T[1]M} \mu Tr\left[\frac{\delta_R S_{BV}}{\delta\underline{A}}\frac{\delta_L\langle\underline{A},\varphi\rangle}{\delta\underline{B}} - (-1)^m \frac{\delta_R S_{BV}}{\delta\underline{B}}\frac{\delta_L\langle\underline{A},\varphi\rangle}{\delta\underline{A}}\right] = \\
& = \int_{T[1]M} \mu Tr[\underline{F}_{\underline{A}}\varphi]
\end{aligned}
$$

then (3.35) holds true. □

Proof of relation (3.36).
As in the previous proof we can consider the following test function

$$
\langle\underline{B}, \psi\rangle = \int_{T[1]M} \mu Tr[\underline{B}\psi] \tag{3.38}
$$

Substituting (3.38) in (3.34) we have

$$
\begin{aligned}
&-\left\{S_{BV}, \langle \underline{B}, \psi\rangle\right\}_{BV} = (-1)^{-(m+1)} \left\langle \left\{S_{BV}, \underline{B}\right\}_{BV}, \psi\right\rangle = \\
&= -\int_{T[1]M} \mu Tr \left[\frac{\delta_R S_{BV}}{\delta \underline{A}} \frac{\delta_L \langle \underline{B}, \psi\rangle}{\delta \underline{B}}\right] = (-1)^{m-2} \int_{T[1]M} \mu Tr[D_{\underline{A}} \underline{B} \psi]
\end{aligned}
$$

then (3.36) holds true. □

(3.34) enjoys the nilpotence property, namely

$$\delta^2_{BV}(\cdot) = 0 \tag{3.39}$$

In the case of the superfields $\underline{A}$ and $\underline{B}$, we have

$$\delta^2_{BV}(\underline{A}) = 0 \tag{3.40}$$

$$\delta^2_{BV}(\underline{B}) = 0 \tag{3.41}$$

Proof of relation (3.40)

$$\delta^2_{BV}(\underline{A}) = \delta_{BV}\delta_{BV}(\underline{A}) = \delta_{BV}\underline{F_{\underline{A}}} = -D_{\underline{A}}\delta_{BV}\underline{A} = -D_{\underline{A}}\underline{F_{\underline{A}}} = 0$$

we used the Bianchi identity (3.33). Then (3.40) holds true. □

Proof of relation (3.41)

$$
\begin{aligned}
\delta_{BV}(\delta_{BV}\underline{B}) &= -\delta_{BV} D_{\underline{A}}\underline{B} = \\
&= (D_{\underline{A}}\delta_{BV}\underline{B} - [\delta_{BV}\underline{A}, \underline{B}]) = +D^2_{\underline{A}}\underline{B} - [\underline{F_{\underline{A}}}, \underline{B}] = 0
\end{aligned}
$$

Where we used the well-known Ricci identity

$$D_{\underline{A}} D_{\underline{A}} \underline{B} = \underline{F_{\underline{A}}}\underline{B} - \underline{B}\underline{F_{\underline{A}}}. \tag{3.42}$$

Then (3.41) holds true. □

The BV action (3.26) is invariant under the BV field variations, namely

$$\delta_{BV} S_{BV} = 0 \tag{3.43}$$

Proof of relation (3.43)

$$
\begin{aligned}
\delta_{BV} S_{BV} = \delta_{BV} k \int_{T[1]M} \mu Tr[\underline{B}\underline{F_{\underline{A}}}] &= k \int_{T[1]M} \mu Tr[\delta_{BV}\underline{B}\underline{F_{\underline{A}}} + \underline{B}\delta_{BV}\underline{F_{\underline{A}}}] = \\
= k \int_{T[1]M} \mu Tr\left[-D_{\underline{A}}\underline{B}\underline{F_{\underline{A}}} + \underline{B} D_{\underline{A}}\underline{F_{\underline{A}}}\right] = 0
\end{aligned}
$$

The last expression vanishes thanks to Bianchi identity (3.33) and Stokes' theorem, therefore equation (3.43) holds true. □

3.2.5 Quantum BV BF theory

In this section we discuss the quantum BV master equation. From the theory of the BV formalism we know that the BV action must satisfy the BV master equation, namely

$$\{S_{BV}, S_{BV}\} - 2i\hbar\Delta_{BV}S_{BV} = 0 \tag{3.44}$$

Since the fact the classical BV equation holds true, we check the Quantum master equation, namely

$$\Delta_{BV}S_{BV} = 0 \tag{3.45}$$

where the BV laplacian has the following form

$$\Delta_{BV} = (-1)^{m+1}\int_{T[1]M} \mu Tr\left(\frac{\delta_L}{\delta \underline{A}}\frac{\delta_L}{\delta \underline{B}}\right) \tag{3.46}$$

Proof of relation (3.45)

$$\begin{aligned}
\Delta S_{BV} &= \Delta_{BV}\ k\int_{T[1]M} \mu Tr(\underline{B}\underline{F}_{\underline{A}}) = \\
&= (-1)^{m+1}k\int_{T[1]M} \mu Tr\left[\frac{\delta_L \underline{F}_{\underline{A}}}{\delta \underline{A}}\right] = \\
&= (-1)^{m+1}k\int_{T[1]M} \mu Tr\left[D_{\underline{A}}\delta(0)1_{\mathfrak{g}}\right] = \\
&= (-1)^{m+1}dim\mathfrak{g}\ \delta(0)\ k\int_{T[1]M} \mu D_{\underline{A}}1 = 0
\end{aligned}$$

where $\delta(0)$ is an infinite constant. In order to have a finite one we impose a cut-off. In an opportune regularization scheme, this result is supposed to be valid before removing the cut-off, in spite of the fact that the Dirac Delta function tends to infinity. □

3.2.6 BV BF theory in 2d

From now, unless stated otherwise, we assume $m = 2$. In this section we study the 2d BV BF model and related gauge fixing procedure.

Geometrical framework

The geometrical framework is constituted by the following elements

(I) An oriented, smooth, compact 2-fold Σ

(II) A principal $G-$bundle P over Σ. Here G is a compact Lie group

We adopt the superfield formalism. The superfield content of this model consists in the following fields

$$\underline{A} - \underline{A_0} \in \underline{\Gamma}(T[1]\Sigma, \mathrm{Ad}P) \tag{3.47}$$

$$\underline{B} \in \underline{\Gamma}(T[1]\Sigma, \mathrm{Ad}P), \tag{3.48}$$

which are respectively a 1-form and a 0-form of adjoint type. We recall that $\underline{A_0}$ is an ordinary background connection of the bundle P which can be viewed as a locally defined field of form-ghost bidegree (1,0).
We can decompose (3.47) and (3.48) in homogeneous components fields as follows

$$\underline{A} - \underline{A_0} = -c + a + b^\dagger \tag{3.49}$$

$$\underline{B} = b + a^\dagger - c^\dagger \tag{3.50}$$

Every component has a $T[1]\Sigma$ degree and $\mathrm{Ad}P$ degree, namely

$\underline{A}$		$\underline{B}$	
c	(0,1)	b	(0,0)
a	(1,0)	$a^\dagger$	(1,-1)
$b^\dagger$	(2,-1)	$c^\dagger$	(2,-2)

In what follows, unless stated otherwise, we assume $\underline{A_0} = 0$.

2d BV action

The BV action for the 2d BF theory is formally

$$S_{BV} = k \int_{T[1]\Sigma} \mu Tr\Big(\underline{B}\ \underline{F_A}\Big) \tag{3.51}$$

we can explicit the action (3.51)on the manifold Σ in field components as follows

$$k \int_\Sigma Tr\left[bF_a - a^\dagger D_a c + b^\dagger[b, c] - c^\dagger cc\right] \tag{3.52}$$

Proof of relation (3.52)
Substituting (3.49) and (3.50) in (3.52) we obtain

$$k \int_\Sigma Tr\left[(b + a^\dagger - c^\dagger)d(-c + a + b^\dagger) + (b + a^\dagger - c^\dagger)(-c + a + b^\dagger)(-c + a + b^\dagger)\right] =$$

$$= k \int_\Sigma Tr\left[bda + baa - a^\dagger(dc + ca + ac) + b^\dagger(bc - cb) - c^\dagger cc\right]$$

Using the usual definitions of curvature and exterior covariant derivative, then (3.52) holds true. □

We have the following BV variations

$$\delta_{BV} a = D_a c \tag{3.53}$$

$$\delta_{BV} b = [b, c] \tag{3.54}$$

$$\delta_{BV} c = cc \tag{3.55}$$

$$\delta_{BV} a^\dagger = D_a b + [c, a^\dagger] \tag{3.56}$$

$$\delta_{BV} b^\dagger = F_a + [c, b^\dagger] \tag{3.57}$$

$$\delta_{BV} c^\dagger = -D_a a^\dagger + [c, c^\dagger] + [b^\dagger, b] \tag{3.58}$$

Gauge fixing for BV BF theory in 2d

We present now the gauge fixing procedure for the 2d BV BF theory. First of all we introduce trivial pairs of fields and antifields, namely

$$\widetilde{c} \in \underline{\Gamma}(\Sigma, \mathrm{Ad}P[1]) \qquad \widetilde{c}^\dagger \in \underline{\Gamma}(\Sigma, \Lambda^2 T^*\Sigma \otimes \mathrm{Ad}P[0]) \tag{3.59}$$

$$\widetilde{\gamma} \in \underline{\Gamma}(\Sigma, \mathrm{Ad}P[0]) \qquad \widetilde{\gamma}^\dagger \in \underline{\Gamma} \in (\Sigma, \Lambda^2 T^*\Sigma) \otimes \mathrm{Ad}P[-1]) \tag{3.60}$$

$$\widetilde{\tau} \in \underline{\Gamma}(\Sigma, \mathrm{Ad}P[0]) \qquad \widetilde{\tau}^\dagger \in \underline{\Gamma}(\Lambda^2 T^*\Sigma \otimes \mathrm{Ad}P[-1]) \tag{3.61}$$

$$\widetilde{\lambda} \in \underline{\Gamma}(\Sigma, \mathrm{Ad}P[1]) \qquad \widetilde{\lambda}^\dagger \in \underline{\Gamma}(\Lambda^2 T^*\Sigma \otimes \mathrm{Ad}P[0]) \tag{3.62}$$

The related BV symplectic form is

$$\Omega_{BVaux} = \int_\Sigma Tr\Big[\delta\widetilde{c}^\dagger \delta\widetilde{c} + \delta\widetilde{\gamma}^\dagger \delta\widetilde{\gamma} + \delta\widetilde{\tau}^\dagger \delta\widetilde{\tau} + \delta\widetilde{\lambda}^\dagger \delta\widetilde{\lambda}\Big] \tag{3.63}$$

The associated BV bracket have the following expression

$$\begin{aligned} \{F, G\}_{BVaux} = \int_\Sigma Tr\Bigg[& \frac{\delta_R F}{\delta\widetilde{c}} \frac{\delta_L G}{\delta\widetilde{c}^\dagger} - \frac{\delta_R F}{\delta\widetilde{c}^\dagger} \frac{\delta_L G}{\delta\widetilde{c}} + \frac{\delta_R F}{\delta\widetilde{\gamma}} \frac{\delta_L G}{\delta\widetilde{\gamma}^\dagger} - \frac{\delta_R F}{\delta\widetilde{\gamma}^\dagger} \frac{\delta_L G}{\delta\widetilde{\gamma}} + \\ & + \frac{\delta_R F}{\delta\widetilde{\tau}} \frac{\delta_L G}{\delta\widetilde{\tau}^\dagger} - \frac{\delta_R F}{\delta\widetilde{\tau}^\dagger} \frac{\delta_L G}{\delta\widetilde{\tau}} + \frac{\delta_R F}{\delta\widetilde{\lambda}} \frac{\delta_L G}{\delta\widetilde{\lambda}^\dagger} - \frac{\delta_R F}{\delta\widetilde{\lambda}^\dagger} \frac{\delta_L G}{\delta\widetilde{\lambda}}\Bigg] \end{aligned} \tag{3.64}$$

Auxiliary BV action

We introduce the auxiliary BV action which reads as follows

$$S_{BVaux} = -\int_\Sigma Tr\left(\widetilde{c}^\dagger \widetilde{\gamma} + \widetilde{\lambda}^\dagger \widetilde{\tau}\right). \tag{3.65}$$

From a direct inspection one has

$$\{S_{BVaux}, S_{BVaux}\}_{BVaux} = 0. \tag{3.66}$$

Equation (3.66) corresponds to the classical master equation for the auxiliary action (3.65).
We define the auxiliary BV variation as follows

$$\delta_{BVaux}(\cdot) = \{S_{BVaux}, \cdot\}_{BVaux}\,. \tag{3.67}$$

Let us calculate auxiliary field variations for fields/antifields introduced in (3.59),(3.60), (3.61) and (3.62) which correspond to

$$\delta_{BVaux}\widetilde{c} = -\widetilde{\gamma} \qquad \delta_{BVaux}\widetilde{c}^{\dagger} = 0 \tag{3.68}$$

$$\delta_{BVaux}\widetilde{\gamma} = 0 \qquad \delta_{BVaux}\widetilde{\gamma}^{\dagger} = -\widetilde{c}^{\dagger} \tag{3.69}$$

$$\delta_{BVaux}\widetilde{\lambda} = -\widetilde{\tau} \qquad \delta_{BVaux}\widetilde{\lambda}^{\dagger} = 0 \tag{3.70}$$

$$\delta_{BVaux}\widetilde{\tau} = 0 \qquad \delta_{BVaux}\widetilde{\tau}^{\dagger} = -\widetilde{\lambda}^{\dagger} \tag{3.71}$$

As expected, the auxiliary BV field variations (3.67) enjoys the nilpotence property, namely

$$\delta^2_{BVaux} = 0. \tag{3.72}$$

Proof of relation (3.72)

$$\begin{aligned}&\delta^2_{BVaux}(\widetilde{c}, \widetilde{c}^{\dagger}, \widetilde{\gamma}, \widetilde{\gamma}^{\dagger}, \widetilde{\lambda}, \widetilde{\lambda}^{\dagger}, \widetilde{\tau}, \widetilde{\tau}^{\dagger}) = \\ &= \delta_{BVaux}\left(-\widetilde{\gamma}, 0, 0, -\widetilde{c}^{\dagger}, -\widetilde{\tau}, 0, 0, -\widetilde{\lambda}^{\dagger}\right) = 0\end{aligned}$$

then (3.72) holds true. □

Auxiliary BV action (3.65) is invariant under auxiliary BV field variations, namely

$$\delta_{BVaux}S_{BVaux} = 0 \tag{3.73}$$

Proof of relation (3.73)

$$\begin{aligned}\delta_{BVaux}S_{BVaux} &= \delta_{BVaux}\left(-\int_{\Sigma} Tr\left[\widetilde{c}^{\dagger}\widetilde{\gamma} + \widetilde{\lambda}^{\dagger}\widetilde{\tau}\right]\right) = \\ &= -\int_{\Sigma} Tr\left[\delta_{BVaux}\widetilde{c}^{\dagger}\widetilde{\gamma} + \widetilde{c}^{\dagger}\delta_{BVaux}\widetilde{\gamma} + \delta_{BVaux}\widetilde{\lambda}^{\dagger}\widetilde{\tau} + \widetilde{\lambda}^{\dagger}\delta_{BVaux}\widetilde{\tau}\right] = 0,\end{aligned}$$

indeed (3.73) holds true. □

The gauge fermion

The gauge fermion for the BV BF theory in 2d is

$$\Psi = \int_{\Sigma} Tr\left(\widetilde{c}D_{A_0} \star a + \widetilde{\tau}D_{A_0} \star a^{\dagger}\right) \tag{3.74}$$

where $\star$, as usual, is the Hodge star.
Using (3.74) we select a lagrangian submanifold $\mathscr{L}$ in the field space as follows

$$\varphi^{\dagger}_{A} = \frac{\delta_L \Psi}{\delta \varphi^A}. \tag{3.75}$$

Then, one finds:

$$a^{\dagger} = \star D_{A_0}\widetilde{c} \tag{3.76}$$
$$b^{\dagger} = 0 \tag{3.77}$$
$$c^{\dagger} = 0 \tag{3.78}$$
$$\widetilde{c}^{\dagger} = D_{A_0} \star a \tag{3.79}$$
$$\widetilde{\lambda}^{\dagger} = 0 \tag{3.80}$$
$$\widetilde{\tau}^{\dagger} = D_{A_0} \star a^{\dagger} \tag{3.81}$$
$$\widetilde{\gamma}^{\dagger} = 0 \tag{3.82}$$

Using the gauge fermion (3.74), we can define the gauge fixed action

$$I = (S_{BV} + S_{BV aux})|_{\mathscr{L}} \tag{3.83}$$

Using (3.52) and (3.65) we have

$$\left\{ k \int_{\Sigma} Tr \left[bF_a - a^{\dagger} D_a c + b^{\dagger}[b,c] - c^{\dagger} cc - \widetilde{c}^{\dagger}\widetilde{\gamma} - \widetilde{\lambda}^{\dagger}\widetilde{\tau} \right] \right\} \Bigg|_{\mathscr{L}} =$$
$$= k \int_{\Sigma} Tr \left[bF_a - \star D_{A_0}\widetilde{c} D_a c - D_{A_0} \star a \widetilde{\gamma} \right]$$

We introduce now the BRST operator, i.e

$$s = \delta_{BV}|_{fields} \tag{3.84}$$

One finds:

$$sa = D_a c \tag{3.85}$$
$$sb = [b,c] \tag{3.86}$$
$$sc = cc \tag{3.87}$$
$$s\widetilde{c} = -\widetilde{\gamma} \tag{3.88}$$
$$s\widetilde{\gamma} = 0 \tag{3.89}$$
$$s\widetilde{\lambda} = -\widetilde{\tau} \tag{3.90}$$
$$s\widetilde{\tau} = 0 \tag{3.91}$$

Operator (3.83) is nilpotent, namely

$$s^2 = 0 \tag{3.92}$$

Proof of relation (3.92)
Consider s^2 acting on $a, b, c, \widetilde{c}, \widetilde{\gamma}, \widetilde{\lambda}, \widetilde{\tau}$, then

$$
\begin{aligned}
& s^2(a+b+c+\widetilde{c}+\widetilde{\gamma}+\widetilde{\lambda}+\widetilde{\tau}) = s(sa+sb+sc+s\widetilde{c}+s\widetilde{\gamma}+s\widetilde{\lambda}+s\widetilde{\tau}) = \\
& = s(+dc+[a,c]+[b,c]+cc-\widetilde{\gamma}-\widetilde{\tau}) = \\
& = -d(sc)+[sa,c]+[a,sc]+[sb,c]+[b,sc]+scc-csc-s\widetilde{\gamma}-s\widetilde{\tau} = \\
& = -d(cc)+dcc-cdc+[[a,c],c]+acc-cca+[[b,c],c]+bcc-ccb+ccc-ccc = 0
\end{aligned}
$$

then the nilpotence property for the BRST operator holds true. □

The gauge fixed action I is invariant under the operator (3.84), i.e.

$$
sI = 0 \tag{3.93}
$$

Proof of relation (3.93)

$$
\begin{aligned}
& s\left(k\int_\Sigma Tr\left[bF_a - \star D_{A_0}\widetilde{c}D_a c - D_{A_0}\star a\widetilde{\gamma}\right]\right) = \\
& = k\int_\Sigma Tr\Big[sbF_a - bsF_a - \star D_{A_0}s\widetilde{c}D_a c - \star D_{A_0}\widetilde{c}sD_a c - D_{A_0}\star sa\widetilde{\gamma}\Big] = \\
& = k\int_\Sigma Tr\Big[[b,c]F_a + b(d(D_a c)+[a,D_a c]) + \star D_{A_0}\widetilde{\gamma}D_a c + \\
& - \star D_{A_0}\widetilde{c}(-d(cc)+[D_a c,c]+[a,cc]) - \star D_{A_0}\widetilde{\gamma}D_a c\Big] = \\
& = k\int_\Sigma Tr\Big[[b,c]F_a + bD_aD_a c\Big] = 0
\end{aligned}
$$

where the last expression vanishes thank to Ricci Identity (3.42). Then (3.93) holds true. □

3.2.7 BV BF theory in 3d

From now, unless stated otherwise, we assume $m = 3$. In this section we provide a BV formulation of the 3d BF model and we also study the gauge fixing procedure.

Geometrical framework

The geometrical framework is constituted by the following data:

(I) An oriented, smooth, compact 3-fold Θ

(II) A principal G-bundle P over Θ. Here G is a compact Lie group

We adopt the superfield formalism. The superfield content in this model consists in the following superfields

$$\underline{A} - \underline{A_0} \in \underline{\Gamma}(T[1]\Theta, \mathrm{Ad}P) \tag{3.94}$$

$$\underline{B} \in \underline{\Gamma}(T[1]\Theta, \mathrm{Ad}P) \tag{3.95}$$

which are both a 1-form of adjoint type.
$\underline{A_0}$ is an ordinary background connection of the bundle P which can be viewed as a locally defined field of form-ghost bidegree (1,0).
We can decompose (3.94) and (3.95) in homogeneous field components as follows

$$\underline{A} = -c + a - b^\dagger - \tau_1^\dagger \tag{3.96}$$

$$\underline{B} = \tau_1 + b + a^\dagger - c^\dagger \tag{3.97}$$

All the components have a definite $T[1]\Theta$ degree and $\mathrm{Ad}P$ degree, namely

$\underline{A}$		$\underline{B}$	
c	(0,1)	τ_1	(0,1)
a	(1,0)	b	(1,0)
$b^\dagger$	(2,-1)	$a^\dagger$	(2,-1)
$\tau_1^\dagger$	(3,-2)	$c^\dagger$	(3,-2)

In what follows, unless stated otherwise, we assume $\underline{A_0} = 0$.

3d BV action

The action of the 3d BV BF theory is formally

$$S_{BV} = k \int_{T[1]\Theta} \mu Tr(\underline{B}\underline{F}_{\underline{A}}) \tag{3.98}$$

We can explicit action (3.98) in field components, i.e.

$$S_{BV} = k \int_{\Theta} Tr \left[bF_a - a^\dagger D_a c \ - \tau_1 D_a b^\dagger + \tau_1 [c, \tau_1^\dagger] + b^\dagger [b, c] - c^\dagger cc \right] \tag{3.99}$$

Proof of equation (3.99).
Substituting (3.96) and (3.97) in (3.98) we obtain

$$
\begin{aligned}
S_{BV} = k \int_\Theta Tr\Big[&(\tau_1 + b + a^\dagger - c^\dagger) d(-c + a - b^\dagger - \tau_1^\dagger) + \\
&+ (\tau_1 + b + a^\dagger - c^\dagger)(-c + a - b^\dagger - \tau_1^\dagger)(-c + a - b^\dagger - \tau_1^\dagger)\Big] = \\
&= k \int_\Theta Tr\Big[- \tau_1 db^\dagger + bda - a^\dagger dc + \tau_1(c\tau_1^\dagger + \tau_1^\dagger c) - \tau_1(ab^\dagger + b^\dagger a) + \\
&+ b(cb^\dagger + b^\dagger c) + baa - a^\dagger(ca + ac) + c^\dagger cc\Big] = \\
&= k \int_\Theta Tr\Big[- \tau_1 db^\dagger + bda - a^\dagger dc + \tau_1 [c, \tau_1^\dagger] - \tau_1 [a, b^\dagger] + \\
&+ b[c, b^\dagger] + baa - a^\dagger[c, a] - c^\dagger cc\Big]
\end{aligned}
$$

Using the standard definitions of curvature and exterior covariant derivative, then (3.99) holds true. □

We have the following BV field variations

$$\delta_{BV} a = D_a c \tag{3.100}$$

$$\delta_{BV} b = D_a \tau_1 + [b, c] \tag{3.101}$$

$$\delta_{BV} c = cc \tag{3.102}$$

$$\delta_{BV} \tau_1 = [\tau_1, c] \tag{3.103}$$

$$\delta_{BV} a^\dagger = D_a b + [c, a^\dagger] + [\tau_1, b^\dagger] \tag{3.104}$$

$$\delta_{BV} b^\dagger = F_a + [b^\dagger, c] \tag{3.105}$$

$$\delta_{BV} c^\dagger = -D_a a^\dagger - [c, c^\dagger] + [b^\dagger, b] - [\tau_1^\dagger, \tau_1] \tag{3.106}$$

$$\delta_{BV} \tau_1^\dagger = [c, \tau_1^\dagger] \tag{3.107}$$

Gauge fixing for the 3d BV BF theory

Now we discuss the gauge fixing procedure for the 3d BV BF theory. First of all is necessary we introduce trivial pairs of fields/antifields, namely

$$\widetilde{c} \in \underline{\Gamma}(\Theta, \mathrm{Ad}P[1]) \qquad \widetilde{c}^\dagger \in \underline{\Gamma}(\Theta, \Lambda^3 T^*\Theta \otimes \mathrm{Ad}P[0]) \tag{3.108}$$

$$\widetilde{\lambda}_1 \in \underline{\Gamma}(\Theta, \mathrm{Ad}P[1]) \qquad \widetilde{\lambda}_1^\dagger \in \underline{\Gamma}(\Theta, \Lambda^3 T^*\Theta \otimes \mathrm{Ad}P[0]) \tag{3.109}$$

$$\widetilde{\gamma} \in \underline{\Gamma}(\Theta, \mathrm{Ad}P[0]) \qquad \widetilde{\gamma}^\dagger \in \underline{\Gamma}(\Theta, \Lambda^3 T^*\Theta \otimes \mathrm{Ad}P[-1]) \tag{3.110}$$

$$\widetilde{\tau}_1 \in \underline{\Gamma}(\Theta, \mathrm{Ad}P[0]) \qquad \widetilde{\tau}_1^\dagger \in \underline{\Gamma}(\Theta, \Lambda^3 T^*\Theta \otimes \mathrm{Ad}P[-1]) \tag{3.111}$$

The related BV symplectic form reads as follows

$$\Omega_{BVaux} = \int_\Theta Tr\Big[\delta\widetilde{c}^\dagger \delta\widetilde{c} + \delta\widetilde{\gamma}^\dagger \delta\widetilde{\gamma} + \delta\widetilde{\tau}_1^\dagger \delta\widetilde{\tau}_1 + +\delta\widetilde{\lambda}_1^\dagger \delta\widetilde{\lambda}_1\Big] \tag{3.112}$$

The associated BV bracket has the following form

$$\{F,G\}_{BVaux}=\int_\Theta Tr\left[\frac{\delta_R F}{\delta\widetilde{c}}\frac{\delta_L G}{\delta\widetilde{c}^\dagger}-\frac{\delta_R F}{\delta\widetilde{c}^\dagger}\frac{\delta_L G}{\delta\widetilde{c}}+\frac{\delta_R F}{\delta\widetilde{\gamma}}\frac{\delta_L G}{\delta\widetilde{\gamma}^\dagger}-\frac{\delta_R F}{\delta\widetilde{\gamma}^\dagger}\frac{\delta_L G}{\delta\widetilde{\gamma}}+\right.$$
$$\left.+\frac{\delta_R F}{\delta\widetilde{\tau}_1}\frac{\delta_L G}{\delta\widetilde{\tau}_1^\dagger}-\frac{\delta_R F}{\delta\widetilde{\tau}_1^\dagger}\frac{\delta_L G}{\delta\widetilde{\tau}_1}+\frac{\delta_R F}{\delta\widetilde{\lambda}_1}\frac{\delta_L G}{\delta\widetilde{\lambda}_1^\dagger}-\frac{\delta_R F}{\delta\widetilde{\lambda}_1^\dagger}\frac{\delta_L G}{\delta\widetilde{\lambda}_1}\right] \tag{3.113}$$

Auxiliary BV action

We can introduce the following auxiliary BV action

$$S_{BVaux}=-\int_\Theta Tr\left(\widetilde{c}^\dagger\widetilde{\gamma}+\widetilde{\lambda}_1^\dagger\widetilde{\tau}_1\right) \tag{3.114}$$

From a direct inspection, one has

$$\{S_{BVaux},S_{BVaux}\}_{BVaux}=0 \tag{3.115}$$

Equation (3.115) corresponds to the classical master equation for the auxiliary action (3.114).
We introduce now the auxiliary BV variations as follows

$$\delta_{BVaux}(\cdot)=\{S_{BVaux},\,.\}_{BVaux} \tag{3.116}$$

Let us calculate the auxiliary BV variations for fields/antifields introduced in (3.108), (3.109),(3.110) and (3.111). We have the following results

$$\delta_{BVaux}\widetilde{c}=-\widetilde{\gamma}\qquad \delta_{BVaux}\widetilde{c}^\dagger=0 \tag{3.117}$$

$$\delta_{BVaux}\widetilde{\gamma}=0\qquad \delta_{BVaux}\widetilde{\gamma}^\dagger=-\widetilde{c}^\dagger \tag{3.118}$$

$$\delta_{BVaux}\widetilde{\lambda}_1=-\widetilde{\tau}_1\qquad \delta_{BVaux}\widetilde{\lambda}_1^\dagger=0 \tag{3.119}$$

$$\delta_{BVaux}\widetilde{\tau}_1=0\qquad \delta_{BVaux}\widetilde{\tau}_1^\dagger=-\widetilde{\lambda}_1^\dagger \tag{3.120}$$

the auxiliary field variations enjoys the nilpotence property, i.e.

$$\delta^2_{BVaux}(\cdot)=0 \tag{3.121}$$

Proof of relation (3.121)

$$\delta^2_{BVaux}\left(\widetilde{c},\widetilde{c}^\dagger,\widetilde{\gamma},\widetilde{\gamma}^\dagger,\widetilde{\lambda}_1,\widetilde{\lambda}_1^\dagger,\widetilde{\tau}_1,\widetilde{\tau}_1^\dagger\right)=$$
$$=\delta_{BVaux}\left(-\widetilde{\gamma},0,0,-\widetilde{c}^\dagger,-\widetilde{\tau}_1,0,0,-\widetilde{\lambda}_1^\dagger\right)=0$$

Then (3.121) holds true.

Auxiliary BV action (3.114) is invariant under auxiliary BV variations, namely

$$\delta_{BVaux}S_{BVaux}=0 \tag{3.122}$$

Proof of relation (3.122)

$$\delta_{BVaux}\left(-\int_\Theta Tr\left(\tilde{c}^\dagger\tilde{\gamma}+\tilde{\lambda}_1^\dagger\tilde{\tau}_1\right)\right)=$$
$$=-\int_\Theta Tr\Big[\delta_{BVaux}\tilde{c}^\dagger\tilde{\gamma}+\tilde{c}^\dagger\delta_{BVaux}\tilde{\gamma}+\delta_{BVaux}\tilde{\lambda}_1^\dagger\tilde{\tau}_1+\tilde{\lambda}_1^\dagger\delta_{BVaux}\tilde{\tau}_1\Big]=0$$

Eq. (3.122) holds true. □

The gauge fermion

The gauge fermion for the 3d BV BF theory is

$$\Psi=\int_\Theta Tr\left(\tilde{c}D_{A_0}\star a+\tilde{\lambda}_1 D_{A_0}\star b\right) \tag{3.123}$$

where $\star$, as usual, is the Hodge star operator.
Using (3.123), we can select a lagrangian submanifold $\mathscr{L}$ in the field space as follows

$$\varphi_A^\dagger=\frac{\delta_L\Psi}{\delta\varphi^A}. \tag{3.124}$$

One finds:

$$a^\dagger=\star D_{A_0}\tilde{c} \tag{3.125}$$
$$b^\dagger=\star D_{A_0}\tilde{\lambda}_1 \tag{3.126}$$
$$c^\dagger=0 \tag{3.127}$$
$$\tau_1^\dagger=0 \tag{3.128}$$
$$\tilde{c}^\dagger=D_{A_0}\star a \tag{3.129}$$
$$\tilde{\lambda}_1^\dagger=D_{A_0}\star b \tag{3.130}$$
$$\tilde{\tau}_1^\dagger=0 \tag{3.131}$$
$$\tilde{\gamma}^\dagger=0 \tag{3.132}$$

Using the gauge fermion (3.123) we can define the gauge fixed action

$$I=(S_{BV}+S_{BVaux})|_{\mathscr{L}}. \tag{3.133}$$

Substituting the relations (3.99) and (3.114), we obtain

$$\Bigg(k\int_\Theta Tr\Big[bF_a-a^\dagger D_a c-\tau_1 D_a b^\dagger+\tau_1[c,\tau_1^\dagger]+$$
$$+b^\dagger[b,c]-c^\dagger cc-\tilde{c}^\dagger\tilde{\gamma}-\tilde{\lambda}_1^\dagger\tilde{\tau}_1\Big]\Bigg)\Bigg|_{\mathscr{L}}=$$

$$=k\int_\Theta Tr\Big(bF_a-\star D_{A_0}\tilde{c}D_a c-\tau_1 D_a\star D_{A_0}\tilde{\lambda}_1+$$
$$+\star D_{A_0}\tilde{\lambda}_1[b,c]-D_{A_0}\star a\tilde{\gamma}-D_{A_0}\star b\tilde{\tau}_1\Big)$$

We can introduce now the BRST operator denoted by s as follows

$$s = \delta_{BV}|_{fields}, \tag{3.134}$$

then, we have

$$\begin{aligned}
sa &= D_a c && (3.135)\\
sb &= D_a\tau_1 + [b,c] && (3.136)\\
sc &= cc && (3.137)\\
s\tau_1 &= [\tau_1, c] && (3.138)\\
s\widetilde{c} &= -\widetilde{\gamma} && (3.139)\\
s\widetilde{\gamma} &= 0 && (3.140)\\
s\widetilde{\lambda}_1 &= -\widetilde{\tau_1} && (3.141)\\
s\widetilde{\tau_1} &= 0 && (3.142)
\end{aligned}$$

Operator (3.134) is nilpotent, i.e.

$$s^2 = 0 \tag{3.143}$$

Proof of relation (3.143).

$$\begin{aligned}
&s\left(sa + sb + sc + s\tau_1 + s\widetilde{c} + s\widetilde{\gamma} + s\widetilde{\lambda}_1 + s\widetilde{\tau_1}\right) = \\
&= s\Big(D_a c + D_a\tau_1 + [b,c] + cc + [\tau_1, c] - \widetilde{\gamma} - \widetilde{\tau_1}\Big) = \\
&= -dcc + cdc + [dc, c] + [[a,c],c] - d([\tau_1, c]) + [dc, \tau_1] + \\
&+ [[a,c],\tau_1] + [a,[\tau_1,c]] + [d\tau_1, c] + [[a,\tau_1],c] + \\
&+ [[b,c],c] + [b,cc] + ccc - ccc + [[\tau_1,c],c] - [\tau_1, cc] = 0
\end{aligned}$$

Therefore (3.143) holds true. □

the gauge fixed action (3.133) is invariant under the BRST operator (3.134), namely

$$sI = 0 \tag{3.144}$$

Proof of equation (3.144)

$$\begin{aligned}
&k\int_\Theta Tr\Big(sbF_a + bsF_a - \star D_{A_0}s\widetilde{c}D_a c + \star D_{A_0}\widetilde{c}sD_a c - sD_a\tau_1 \star D_{A_0}\widetilde{\lambda}_1 + \\
&+ D_a\tau_1 \star D_{A_0}s\widetilde{\lambda}_1 - D_{A_0}s\widetilde{\lambda}_1[b,c] + \star D_{A_0}\widetilde{\lambda}_1([sb,c] + [b,sc]) - \star D_{A_0}sa\widetilde{\gamma} + \\
&- \star D_{A_0}as\widetilde{\gamma} + \star D_{A_0}sb\widetilde{\tau_1} - \star D_{A_0}bs\widetilde{\tau_1}\Big) = \\
&= k\int_\Theta Tr\Big((D_a\tau_1 + [b,c])F_a + b(d(D_a c) + [D_a c, a]) + (-d(cc) + [D_a c, c] + \\
&+ [a, cc]) \star D_{A_0}\widetilde{c} + D_a c \star D_{A_0}\widetilde{\gamma} + (-d([\tau_1,c]) + [D_a c, \tau_1] + [a,[\tau_1,c]]) \star D_{A_0}\widetilde{\lambda}_1 + \\
&+ D_a\tau_1 \star D_{A_0}\widetilde{\tau_1} + \star D_{A_0}\widetilde{\tau_1}[b,c] + \star D_{A_0}\widetilde{\lambda}_1([D_a\tau_1, c] + [[b,c],c] + [b,cc]) + \\
&- D_a c \star D_{A_0}\widetilde{\gamma} - \star D_{A_0}\widetilde{\tau_1}(D_a\tau_1 + [b,c])\Big) = \\
&= k\int_\Theta Tr\Big((D_a\tau_1 + [b,c])F_a + bD_aD_a c\Big) = 0
\end{aligned}$$

where the last expression vanishes thanks to Ricci identity and Stokes' theorem. Then (3.144) holds true.

3.2.8 BV BF theory in 4d

From now, unless stated otherwise, we assume $m = 4$. In this section we provide a BV formulation of the 4d BF model and we also study the gauge fixing procedure.

Geometrical Framework

The geometrical framework for this model is constituted by the following data:

(I) An oriented, smooth, compact 4-fold Π

(II) A principal G bundle P over Π. Here G is a compact Lie group.

We adopt the superfield formalism. The superfield content of this model consists in the following superfields

$$\underline{A} - \underline{A_0} \in \underline{\Gamma}(T[1]\Pi, \mathrm{Ad}P) \tag{3.145}$$

$$\underline{B} \in \underline{\Gamma}(T[1]\Pi, \mathrm{Ad}P) \tag{3.146}$$

which are respectively a 1-form and a 2-form of adjoint type.
We recall that $\underline{A_0}$ is an ordinary background connection of the bundle P which can be viewed as a locally defined field of form- ghost bidegree $(1, 0)$.
We can decompose (3.145) and (3.146) in homogeneous component fields as follows

$$\underline{A} - \underline{A_0} = -c + a + b^\dagger + \tau_1^\dagger + \tau_2^\dagger \tag{3.147}$$

$$\underline{B} = \tau_2 + \tau_1 + b + a^\dagger - c^\dagger \tag{3.148}$$

every component has a definite $T[1]\Pi$ degree and a $\mathrm{Ad}P$ degree, namely

$\underline{A}$		$\underline{B}$	
c	(0,1)	τ_2	(0,2)
a	(1,0)	τ_1	(1,1)
$b^\dagger$	(2,-1)	b	(2,0)
$\tau_1^\dagger$	(3,-2)	$a^\dagger$	(3,-1)
$\tau_2^\dagger$	(4,-3)	$c^\dagger$	(4,-2)

In what follows, unless stated otherwise, we assume $\underline{A_0} = 0$.

4d BV action

The action for the 4d BV BF theory is formally

$$S_{BV} = k \int_{T[1]\Pi} \mu Tr \left(\underline{B F_{\underline{A}}} \right) \tag{3.149}$$

We can express action (3.149) in field components as follows

$$\begin{aligned} S_{BV} = k \int_\Pi Tr \Big(& \tau_2 D_a \tau_1^\dagger + \tau_1 D_a b^\dagger - a^\dagger D_a c + b F_a + \tau_2 b^\dagger b^\dagger + \\ & - \tau_1 [c, \tau_1^\dagger] - b[c, b^\dagger] - \tau_2 [c, \tau_2^\dagger] - c^\dagger c c \Big) \end{aligned} \tag{3.150}$$

Proof of relation (3.150).
Using (3.147) and (3.148) in (3.149) we obtain

$$
\begin{aligned}
&k\int_\Pi Tr\Big[\left(\tau_2+\tau_1+b+a^\dagger-c^\dagger\right)d\left(-c+a+b^\dagger+\tau_1^\dagger+\tau_2^\dagger\right)+\\
&+\left(\tau_2+\tau_1+b+a^\dagger-c^\dagger\right)\left(-c+a+b^\dagger+\tau_1^\dagger+\tau_2^\dagger\right)\left(-c+a+b^\dagger+\tau_1^\dagger+\tau_2^\dagger\right)\Big]=\\
&=k\int_\Pi Tr\Big[\tau_2\Big(d\tau_1^\dagger+[a,\tau_1^\dagger]+b^\dagger b^\dagger-[c,\tau_2^\dagger]\Big)+b\Big(F_a-[c,b^\dagger]\Big)+\\
&+\tau_1\Big(db^\dagger-[c,\tau_1^\dagger]+[a,b^\dagger]\Big)-a^\dagger\Big(dc+[a,c]\Big)-c^\dagger cc\Big]
\end{aligned}
$$

Using the standard definitions of exterior covariant derivative and curvature (3.150) holds true.

We have the following BV field variations, namely

$$\delta_{BV}a=D_ac \tag{3.151}$$

$$\delta_{BV}b=D_a\tau_1+[b,c]+[\tau_2,b^\dagger] \tag{3.152}$$

$$\delta_{BV}c=cc \tag{3.153}$$

$$\delta_{BV}\tau_1=D_a\tau_2-[\tau_1,c] \tag{3.154}$$

$$\delta_{BV}\tau_2=[\tau_2,c] \tag{3.155}$$

$$\delta_{BV}a^\dagger=[\tau_1^\dagger,\tau_2]-[b^\dagger,\tau_1+[a^\dagger,c]-D_ab \tag{3.156}$$

$$\delta_{BV}b^\dagger=F_a-[c,b^\dagger] \tag{3.157}$$

$$
\begin{aligned}
\delta_{BV}c^\dagger=&[c,c^\dagger]-D_aa^\dagger+[\tau_1,\tau_1^\dagger]+\\
&-[b^\dagger,b]+[\tau_2^\dagger,\tau_2]+[b^\dagger,b]
\end{aligned}
\tag{3.158}
$$

$$\delta_{BV}\tau_1^\dagger=D_ab^\dagger+[c,\tau_1^\dagger] \tag{3.159}$$

$$\delta_{BV}\tau_2^\dagger=D_a\tau_1^\dagger-[c,\tau_2^\dagger]+b^\dagger b^\dagger \tag{3.160}$$

Gauge fixing for the 4d BV BF theory

We discuss the gauge fixing procedure for the 4d BV BF theory. First of all we introduce trivial pairs of fields/antifields, namely

$$\widetilde{c}\in\underline{\Gamma}(\Pi,\mathrm{Ad}P[1])\qquad \widetilde{c}^\dagger\in\underline{\Gamma}(\Pi,\Lambda^4T^*\Pi\otimes\mathrm{Ad}P[0]) \tag{3.161}$$

$$\widetilde{\lambda}_1\in\underline{\Gamma}(\Pi,\mathrm{Ad}P[1])\qquad \widetilde{\lambda}_1^\dagger\in\underline{\Gamma}(\Pi,\Lambda^4T^*\Pi\otimes\mathrm{Ad}P[0]) \tag{3.162}$$

$$\widetilde{\lambda}_2\in\underline{\Gamma}(\Pi,\mathrm{Ad}P[2])\qquad \widetilde{\lambda}_2^\dagger\in\underline{\Gamma}(\Pi,\Lambda^4T^*\Pi\otimes\mathrm{Ad}P[1]) \tag{3.163}$$

$$\widetilde{\gamma}\in\underline{\Gamma}(\Pi,\mathrm{Ad}P[0])\qquad \widetilde{\gamma}^\dagger\in\underline{\Gamma}(\Pi,\Lambda^4T^*\Pi\otimes\mathrm{Ad}P[-1]) \tag{3.164}$$

$$\widetilde{\tau}_1\in\underline{\Gamma}(\Pi,\mathrm{Ad}P[0])\qquad \widetilde{\tau}_1^\dagger\in\underline{\Gamma}(\Pi,\Lambda^4T^*\Pi\otimes\mathrm{Ad}P[-1]) \tag{3.165}$$

$$\widetilde{\tau}_2\in\underline{\Gamma}(\Pi,\mathrm{Ad}P[-1])\qquad \widetilde{\tau}_2^\dagger\in\underline{\Gamma}(\Pi,\Lambda^4T^*\Pi\otimes\mathrm{Ad}P[-2]) \tag{3.166}$$

The auxiliary BV symplectic form this model is

$$\Omega_{BVaux} = \int_{\Pi} Tr\Big(\delta\widetilde{c}^{\dagger}\delta\widetilde{c} + \delta\widetilde{\gamma}^{\dagger}\delta\widetilde{\gamma} + \delta\widetilde{\lambda_1}^{\dagger}\delta\widetilde{\lambda_1} + \\ + \delta\widetilde{\tau_1}^{\dagger}\delta\widetilde{\tau_1} + \delta\widetilde{\lambda_2}^{\dagger}\delta\widetilde{\lambda_2} + \delta\widetilde{\tau_2}^{\dagger}\delta\widetilde{\tau_2}\Big) \tag{3.167}$$

while the associated BV bracket reads as follows

$$\{F,G\}_{BVaux} = \int_{\Pi} Tr\Bigg[\frac{\delta_R F}{\delta\widetilde{c}}\frac{\delta G}{\delta\widetilde{c}^{\dagger}} - \frac{\delta_R F}{\delta\widetilde{c}^{\dagger}}\frac{\delta_L G}{\delta\widetilde{c}} + \frac{\delta_R F}{\delta\widetilde{\gamma}}\frac{\delta G}{\delta\widetilde{\gamma}^{\dagger}} - \frac{\delta_R F}{\delta\widetilde{\gamma}^{\dagger}}\frac{\delta_L G}{\delta\widetilde{\gamma}} + \\ + \frac{\delta_R F}{\delta\widetilde{\lambda_1}}\frac{\delta G}{\delta\widetilde{\lambda_1}^{\dagger}} - \frac{\delta_R F}{\delta\widetilde{\lambda_1}^{\dagger}}\frac{\delta_L G}{\delta\widetilde{\lambda_1}} + \frac{\delta_R F}{\delta\widetilde{\tau_1}}\frac{\delta G}{\delta\widetilde{\tau_1}^{\dagger}} - \frac{\delta_R F}{\delta\widetilde{\tau_1}^{\dagger}}\frac{\delta_L G}{\delta\widetilde{\tau_1}} + \\ + \frac{\delta_R F}{\delta\widetilde{\lambda_2}}\frac{\delta G}{\delta\widetilde{\lambda_2}^{\dagger}} - \frac{\delta_R F}{\delta\widetilde{\lambda_2}^{\dagger}}\frac{\delta_L G}{\delta\widetilde{\lambda_2}} + \frac{\delta_R F}{\delta\widetilde{\tau_2}}\frac{\delta G}{\delta\widetilde{\tau_2}^{\dagger}} - \frac{\delta_R F}{\delta\widetilde{\tau_2}^{\dagger}}\frac{\delta_L G}{\delta\widetilde{\tau_2}}\Bigg] \tag{3.168}$$

Auxiliary BV action

We can introduce the auxiliary BV action as follows

$$S_{BVaux} = -\int_{\Pi} Tr\left(\widetilde{c}^{\dagger}\widetilde{\gamma} + \widetilde{\lambda_1}^{\dagger}\widetilde{\tau_1} + \widetilde{\lambda_2}^{\dagger}\widetilde{\tau_2}\right) \tag{3.169}$$

From a direct inspection the following relation holds true

$$\{S_{BVaux}, S_{BVaux}\}_{BVaux} = 0 \tag{3.170}$$

Equation (3.170) corresponds to the classical master equation for the auxiliary action (3.169).

We introduce now the auxiliary BV variations as follows

$$\delta_{BVaux}(\cdot) = \{S_{BVaux}, \cdot\}_{BVaux} \tag{3.171}$$

Let us calculate the auxiliary BV variations for the fields and antifields introduced previously. We have the following results

$$\delta_{BVaux}\widetilde{c} = -\widetilde{\gamma} \qquad \delta_{BVaux}\widetilde{c}^{\dagger} = 0 \tag{3.172}$$

$$\delta_{BVaux}\widetilde{\gamma} = 0 \qquad \delta_{BVaux}\widetilde{\gamma}^{\dagger} = -\widetilde{c}^{\dagger} \tag{3.173}$$

$$\delta_{BVaux}\widetilde{\lambda_1} = -\widetilde{\tau_1} \qquad \delta_{BVaux}\widetilde{\lambda_1}^{\dagger} = 0 \tag{3.174}$$

$$\delta_{BVaux}\widetilde{\tau_1} = 0 \qquad \delta_{BVaux}\widetilde{\tau_1}^{\dagger} = -\widetilde{\lambda_1}^{\dagger} \tag{3.175}$$

$$\delta_{BVaux}\widetilde{\lambda_2} = -\widetilde{\tau_2} \qquad \delta_{BVaux}\widetilde{\lambda_2}^{\dagger} = 0 \tag{3.176}$$

$$\delta_{BVaux}\widetilde{\tau_2} = 0 \qquad \delta_{BVaux}\widetilde{\tau_2}^{\dagger} = -\widetilde{\lambda_2}^{\dagger} \tag{3.177}$$

As expected, we have the nilpotence property for the auxiliary field variations (3.171), i.e.

$$\delta^2_{BVaux}(\cdot) = 0 \tag{3.178}$$

Proof of relation (3.178)

$$\delta^2_{BVaux}\left(\widetilde{c}, \widetilde{c}^\dagger, \widetilde{\gamma}, \widetilde{\gamma}^\dagger, \widetilde{\lambda}_1, \widetilde{\lambda}_1^\dagger, \widetilde{\tau}_1, \widetilde{\tau}_1^\dagger, \widetilde{\lambda}_2, \widetilde{\lambda}_2^\dagger, \widetilde{\tau}_2, \widetilde{\tau}_2^\dagger\right) =$$

$$= \delta_{BVaux}\left(-\widetilde{\gamma}, 0, 0, -\widetilde{c}^\dagger, -\widetilde{\tau}_1, 0, 0, -\widetilde{\lambda}_1^\dagger, -\widetilde{\tau}_2, 0, 0, -\widetilde{\lambda}_2^\dagger\right) = 0$$

Then relation (3.178) holds true. □

Auxiliary BV action (3.169) enjoys the property of invariance under Auxiliary BV variations (3.171) , namely

$$\delta_{BVaux} S_{BVaux} = 0 \tag{3.179}$$

Proof of relation (3.179)

$$\delta_{BVaux}\left(-\int_\Pi Tr\left(\widetilde{c}^\dagger\widetilde{\gamma} + \widetilde{\lambda}_1^\dagger\widetilde{\tau}_1 + \widetilde{\lambda}_2^\dagger\widetilde{\tau}_2\right)\right)$$

applying the variations on the fields and antifields we have

$$-\int_\Pi Tr\Big(\delta_{BVaux}\widetilde{c}^\dagger\widetilde{\gamma} + \widetilde{c}^\dagger\delta_{BVaux}\widetilde{\gamma} + \delta_{BVaux}\widetilde{\lambda}_1^\dagger\widetilde{\tau}_1 +$$

$$+ \widetilde{\lambda}_1^\dagger\delta_{BVaux}\widetilde{\tau}_1 + \delta_{BVaux}\widetilde{\lambda}_2^\dagger\widetilde{\tau}_2 + \widetilde{\lambda}_2^\dagger\delta_{BVaux}\widetilde{\tau}_2\Big) = 0$$

then (3.179) holds true. □

The gauge fermion

The gauge fermion for the 4d BV BF theory is formally

$$\Psi = \int_\Pi Tr\left(\widetilde{c}D_{A_0} \star a + \widetilde{\lambda}_1 D_{A_0} \star b + \widetilde{\lambda}_2 D_{A_0} \star \tau_1\right), \tag{3.180}$$

where $\star$ denotes, as usual, the Hodge star operator.
Using (3.180), we can select a lagrangian submanifold $\mathscr{L}$ in the field space as follows

$$\varphi_A^\dagger = \frac{\delta_L \Psi}{\delta\varphi^A} \tag{3.181}$$

One finds:

$$a^\dagger = \star D_{A_0}\widetilde{c} \tag{3.182}$$

$$b^\dagger = \star D_{A_0}\widetilde{\lambda}_1 \tag{3.183}$$

$$c^\dagger = 0 \tag{3.184}$$

$$\tau_1^\dagger = \star D_{A_0}\widetilde{\lambda}_2 \tag{3.185}$$

$$\tau_2^\dagger = 0 \tag{3.186}$$

$$\widetilde{c}^\dagger = D_{A_0} \star a \tag{3.187}$$

$$\widetilde{\gamma}^\dagger = 0 \tag{3.188}$$

$$\widetilde{\lambda_1}^\dagger = D_{A_0} \star b \tag{3.189}$$

$$\widetilde{\lambda_2}^\dagger = D_{A_0} \star \tau_1 \tag{3.190}$$

$$\widetilde{\tau_1}^\dagger = 0 \tag{3.191}$$

$$\widetilde{\tau_2}^\dagger = 0 \tag{3.192}$$

Using the gauge fermion (3.180) we can define the gauge fixed action, namely

$$I = (S_{BV} + S_{BVaux})\,|_{\mathscr{L}} \tag{3.193}$$

Using (3.150) and (3.169) in (3.193), then we have

$$\begin{aligned}
I = &\Bigg(k\int_\Pi Tr\Big(\tau_2 D_a \tau_1^\dagger + \tau_1 D_a b^\dagger - a^\dagger D_a c + bF_a + \tau_2 b^\dagger b^\dagger + \tau_1^\dagger[\tau_1, c] + \\
&+ b^\dagger[b,c] - \tau_2[c,\tau_2^\dagger] - c^\dagger cc - \widetilde{c}^\dagger\widetilde{\gamma} - \widetilde{\lambda_1}^\dagger\widetilde{\tau_1} - \widetilde{\lambda_2}^\dagger\widetilde{\tau_2}\Big)\Bigg)\Bigg|_{\mathscr{L}} = \\
&= k\int_\Pi Tr\Big(\tau_2 D_a \star D_{A_0}\widetilde{\lambda_2} + \tau_1 D_a \star D_{A_0}\widetilde{\lambda_1} - \star D_{A_0}\widetilde{c} D_a c + bF_a + \\
&+ \tau_2 \star D_{A_0}\widetilde{\lambda_1} \star D_{A_0}\widetilde{\lambda_1} + \star D_{A_0}\widetilde{\lambda_2}[\tau_1, c] + \star D_{A_0}\widetilde{\lambda_1}[b,c] + \\
&- D_{A_0} \star a\widetilde{\gamma} - D_{A_0} \star b\widetilde{\tau_1} - D_{A_0} \star \tau_1\widetilde{\tau_2}\Big)
\end{aligned}$$

We can introduce now the BRST operator as follows

$$s = \delta_{BV}\Big|_{fields}, \tag{3.194}$$

then we have

$$sa = D_a c \tag{3.195}$$

$$sb = D_a \tau_1 + [b,c] \tag{3.196}$$

$$sc = cc \tag{3.197}$$

$$s\tau_1 = D_a \tau_2 - [\tau_1, c] \tag{3.198}$$

$$s\tau_2 = [\tau_2, c] \tag{3.199}$$

$$s\widetilde{c} = -\widetilde{\gamma} \tag{3.200}$$

$$s\widetilde{\gamma} = 0 \tag{3.201}$$

$$s\widetilde{\lambda_1} = -\widetilde{\tau_1} \tag{3.202}$$

$$s\widetilde{\lambda_2} = -\widetilde{\tau_2} \tag{3.203}$$

$$s\widetilde{\tau_1} = 0 \tag{3.204}$$

$$s\widetilde{\tau_2} = 0 \tag{3.205}$$

The BRST operator (3.194) is nilpotent only on shell, namely, for the field b, we have

$$s^2 b = [F_a, \tau_2] \neq 0 \tag{3.206}$$

Proof of relation (3.206)
Using relation (3.196) we have

$$\begin{aligned} s^2 b &= s(D_a\tau_1 + [b,c]) = s(d\tau_1) + [a,\tau_1] + [b,c]) = \\ &= -d(d\tau_2 + [a,\tau_2]) + d([\tau_1,c]) + [dc,\tau_1] + [[a,c],\tau_1] + [a,d\tau_2] + \\ &+ [a,[a,\tau_2]] - [a,[\tau_1,c]] + [d\tau_1,c] + [[a,\tau_1],c] + [[b,c],c] + [b,cc] = \\ &= \tau_2(da + 2[a,a]) - (da + 2[a,a])\tau_2 = [\tau_2, F_a] \neq 0 \end{aligned}$$

then relation (3.206) holds true. □

while, for the other fields, we have

$$s^2 = 0 \tag{3.207}$$

Proof of relation (3.207)
We consider s^2 acting on the fields a, c, τ_1 and τ_2, then

$$\begin{aligned} & s^2(a + c + \tau_1 + \tau_2) = s\left(D_a c + cc + D_a\tau_2 - [\tau_1,c] + [\tau_2,c]\right) = \\ & = -d(cc) + [D_a c, c] + [a,cc] + ccc - ccc + d([\tau_2,c]) + [D_a c,\tau_2] + \\ & + [a,[\tau_2,c]] - [D_a\tau_2,c] + [[\tau_1,c],c] + [\tau_1,cc] + [[\tau_2,c],c] + [\tau_2,cc] = 0 \end{aligned}$$

then (3.207) holds true. □

Chapter 4

Poisson Sigma Model

In this chapter we present a very important example of topological field theory: The Poisson Sigma Model. It was introduced by Noriaki Ikeda in [20] and later Strobl and Schaller unified several models of gravity and they recast them in a common form in Yang-Mills theory [28]. Cattaneo and Felder in [8] provided a BV formulation of this model. The Poisson Sigma model consists in a bi-dimensional field theory defined on a manifold possibly with boundary. In the first section we provide a description of the ordinary Poisson sigma model studying its action and related symmetries and applying the BRST formalism on it. Then, in the second section, we introduce the BV version of the same model in order to study the gauge-fixing procedure.

4.1 Ordinary Poisson Sigma Model

4.1.1 Geometrical framework

We present now the essential geometrical elements necessary to introduce the classical action of this model.

Let M be a Poisson manifold which is a smooth, paracompact, finite-dimensional manifold endowed with a Poisson structure. We can define a structure like that considering a bivector $\alpha \in C^\infty(M, \Lambda^2 T[1]M)$ that can be seen as contravariant tensor α of rank 2, satisfying the following Jacobi Identity

$$\alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0 \tag{4.1}$$

or shortly

$$\alpha^{i[j}\partial_i\alpha^{kl]} = 0 \tag{4.2}$$

Using the Poisson bivector field, we can define the Poisson Bracket as follows

$$\{f, g\} = \alpha^{ij}\partial_i f \partial_j g, \tag{4.3}$$

where f and g are smooth functions. (4.3) enjoys the following property:

-Antisymmetry

$$\{f, g\} + \{g, f\} = 0 \tag{4.4}$$

-Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0 \tag{4.5}$$

-Leibnitz rule

$$\{f,gh\}=\{f,g\}\,h+\{f,h\}\,g \tag{4.6}$$

The Poisson Sigma model is a sigma model whose worldsheet is a connected, smooth bidimensional manifold Σ (with boundaries) and whose target is a (m-dimensional) Poisson manifold (M,α).

Remark 4.1 Note that the space of functions $\mathscr{F}(M)$ defined on a manifold M endowed with a Poisson structure $\{\cdot,\cdot\}$ is an infinite-dimensional Lie algebra.

4.1.2 Classical Action and symmetries

Action

The field content of this model is constituted by the following elements:

(i) A bosonic field $X:\Sigma\longrightarrow M$, also called embedding field

(ii) A bosonic field η which is a 1-form on Σ taking values in the pull-back by X of the cotangent bundle of M

Using the previous fields we can define the classical action as follows

$$S(X,\eta)=\int_\Sigma \langle\eta,dX\rangle+\frac{1}{2}\langle\eta,(\alpha\circ X)\eta\rangle\,, \tag{4.7}$$

where we denoted with $\langle\cdot,\cdot\rangle$ the pairing between the cotangent and tangent space at a point of the manifold M.
The eventually boundary conditions for (4.7) are the following

$$\eta(v)u=0 \qquad v\in\partial\Sigma \qquad u\in T_v(\partial\Sigma) \tag{4.8}$$

The two bosonic fields can be expressed in local coordinates as follows:

(i) X is given by m functions $X^i(v)$

(ii) η is given by d differential 1-forms $\eta_i(v)=\eta_{i,\mu}(v)dv^\mu$

Using the previous expressions for the fields we can express action (4.7) in the following way, viz.

$$S(X,\eta)=\int_\Sigma \eta_i dX^i+\frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j. \tag{4.9}$$

In the case of a manifold with boundaries we have for η, $v\in\partial\Sigma$, while for $\eta_i(v)$ vanishes on tangent vectors to $\partial\Sigma$.

Let us calculate the field equations. Using the Euler-Lagrange equation we obtain

$$\frac{\delta S(X,\eta)}{\delta \eta_i} = dX^i + \alpha^{ij}(X)\eta_j \tag{4.10}$$

and

$$\frac{\delta S(X,\eta)}{\delta X^i} = d\eta_i + \frac{1}{2}\partial_i \alpha^{kj}(X)\eta_k\eta_j \tag{4.11}$$

Symmetries

The action (4.9) is invariant under the following infinitesimal gauge transformations

$$\delta_\beta X^i = \alpha^{ij}(X)\beta_j \tag{4.12}$$

$$\delta_\beta \eta_i = -d\beta_j - \partial_i \alpha^{jk}(X)\eta_j\beta_k \tag{4.13}$$

where β_i is an infinitesimal parameter, which is a section of $X^*(T^*[1]M)$ and vanishes on the (eventually) boundary conditions of the manifold.
We have a special case when we impose the condition $\alpha = 0$, so the action (4.9) reads as follows

$$S(x,\eta) = \int_\Sigma \eta_i(v) \wedge dX^i(v). \tag{4.14}$$

The action (4.14) is invariant under the following transformations

$$\eta \longrightarrow \eta + \widetilde{\eta}, \tag{4.15}$$

where $\widetilde{\eta}$ is an exact one-forms on Σ.

Another important symmetry arises when we consider α^{ij} as an invertible matrix. In this case after integrating formally over the field η we recover the action

$$S(X) = \int_\Sigma X^*\omega = \frac{1}{2}\int_\Sigma \omega_{ij}(x)dx^i dx^j \tag{4.16}$$

where ω is the symplectic form which generates the bivector α. In this peculiar case the differential condition (4.1) implies that ω^{ij} is a closed 2-form. Now action (4.16) in invariant under the following transformations

$$X^i \longrightarrow X^i + \xi^i \tag{4.17}$$

with $\xi^i(v) = 0$ on the eventually boundary of the manifold Σ.

Proof of the symmetry of the action under the transformations (4.17)

$$\begin{aligned}\delta S = \delta \int_\Sigma X^*\omega &= \int_\Sigma \frac{1}{2}\partial_k\omega_{ij}\xi^k dx^i dx^j + d\left(\omega_{ij}\xi^i dx^j\right) - \partial_i\omega_k dx^i \xi^k = \\ &= \frac{1}{2}\int_\Sigma \left[\partial_k\omega_{ij} + \partial_i\omega_{jk} + \partial_j\omega_{ik}\right]\xi^k dx^i dx^j = \\ &= \frac{1}{2}\int_\Sigma (d\omega)_{ijk}\xi^k dx^i dx^j = 0\end{aligned}$$

Then the action is invariant under the transformation (4.17). □

We can assume now M as a vector space and α a linear function on M. In this case M is the dual space to a Lie algebra $\mathfrak{g} = M^*$. Given two linear functions $f, g \in M^*$ we can construct the Lie bracket which corresponds to the ordinary Poisson bracket whose is again a linear function on M.
The classical action (4.9) can be viewed as a function of a field X taking values in the adjoint lie algebra of $\mathfrak{g}$ and a connection $d + \eta$ on a principal bundle on Σ which we assume to be trivial.
Integrating by parts, the action (4.9) can be written in the same form of the BF theory action (cit), namely

$$S = \int_\Sigma \langle X, F_\eta \rangle = \int_\Sigma X d\eta + X(\eta \wedge \eta) \tag{4.18}$$

4.1.3 BRST formalism

We can apply the BRST formalism to the Poisson sigma model promoting the infinitesimal parameter β_i to an anticomutting ghost field (with appropriate boundary conditions). We can introduce the BRST operator s, which is an odd derivation on the functions X, η, β, such that the following relations hold true

$$sX^i = \alpha^{ij}\beta_j \tag{4.19}$$

$$s\eta_i = -d\beta_i - \partial_i \alpha^{kl}(X)\eta_k\beta_l \tag{4.20}$$

$$s\beta_i = \frac{1}{2}\partial_i \alpha^{jk}(X)\beta_j\beta_k \tag{4.21}$$

The BRST operator is nilpotent only on shell, namely

$$s^2 X^i = 0 \tag{4.22}$$

$$s^2 \beta_i = 0 \tag{4.23}$$

$$s^2 \eta_i = -\frac{1}{2}\partial_i\partial_k \alpha^{qs}\beta_q\beta_s(dX^k + \alpha^{kj}(X)\eta_j) \tag{4.24}$$

We recognize in (4.24) the field equation (4.10).

We assign a ghost number to the fields. Considering also the form degree we have the following form-ghost bidegrees for the fields

$$\begin{array}{ll} X^i & (0,0) \\ \beta_i & (0,1) \\ \eta_i & (1,0) \end{array}$$

The BRST operator has ghost number 1.
We have already seen that s squares to zero only on shell. A problem arises with this formalism. When we try to calculate the path integral to determine a physical observables this does not work because we do not have a well-defined cohomology. In order to solve this problem and to quantize this model we use the Batalin-Vilkovisky formalism, that is a generalization of the BRST ones.

4.2 BV Poisson Sigma Model

In this section we present a BV formulation of the Poisson sigma model presented in the previous section. We also study the gauge fixing procedure in order to obtain the quantum theory for this model.

4.2.1 Geometrical framework and superfield formalism

The geometrical data for this model are the following

I An oriented bidimensional manifold Σ (possibly with boundaries)

II A space of bundle maps from $T[1]\Sigma$ to $T^*[+1]M$ of a Poisson manifold M. We denote such bundle map by pair (X,η), where $X:\Sigma \longrightarrow M$ is the base map and η is the map between fibers, is a section in $\Gamma(\Sigma, Hom(T[1]\Sigma), X^*(T^*[+1]M))$

In this model we adopt the superfield formalism. Superfields combine fields and antifields. The superfields content in this model consist in a total degree zero superfield $\widetilde{X}^i$, which can be decomposed as follows:

$$\widetilde{X}^i = X^i + \theta^\mu \eta^{i\dagger}_\mu - \frac{1}{2}\theta^\mu\theta^\nu\beta^{i\dagger}_{\mu\nu}. \tag{4.25}$$

We also have a total degree one odd superfield $\widetilde{\eta}_i$, which can be decomposed as follows:

$$\widetilde{\eta}_i = \beta_i + \theta^\mu \eta_{i,\mu} + \frac{1}{2}\theta^\mu\theta^\nu X^\dagger_{i,\mu,\nu}. \tag{4.26}$$

We recall shortly the integration of superfields. We can perform this integration using the standard supermeasure μ of $T[1]\Sigma$, μ has $T[1]\Sigma$ degree -2.
If φ is a superfield, one has

$$\int_{T[1]\Sigma} \mu\varphi = \int_\Sigma \varphi^{(2)}, \tag{4.27}$$

where $\varphi^{(2)}$ is the component of $T[1]\Sigma$ which has standard form degree 2.

4.2.2 BV symplectic form

In this theory is relevant the following symplectic form

$$\Omega_{BV} = \int_{T[1]\Sigma} \mu\delta\widetilde{\eta}_i\delta\widetilde{X}^i \tag{4.28}$$

The BV symplectic form (4.28) has degree -1 and is closed as required

$$\delta\Omega_{BV} = 0 \tag{4.29}$$

The associated BV bracket for this model reads as follows

$$\{F,G\} = \int_{T[1]\Sigma} \mu \left[\frac{\delta_R F}{\delta\widetilde{X}^i}\frac{\delta_L G}{\delta\widetilde{\eta}_i} - \frac{\delta_R F}{\delta\widetilde{\eta}_i}\frac{\delta_L G}{\delta\widetilde{X}^i}\right] \tag{4.30}$$

Using (4.25) and (4.26) we can express (4.28) in field components as follows

$$\Omega_{BV} = \int_{\Sigma} \left(\delta\beta^{i\dagger}\delta\beta_i + \delta\eta^{i\dagger}\delta\eta_i + \delta X_i^{\dagger}\delta X^i \right) \tag{4.31}$$

The associated BV bracket in field components has the following form

$$\begin{aligned}\{F, G\} = \int_{\Sigma} \Bigg[& \frac{\delta_R F}{\delta\beta_i}\frac{\delta_L G}{\delta\beta^{i\dagger}} - \frac{\delta_R F}{\delta\beta^{i\dagger}}\frac{\delta_L G}{\delta\beta_i} + \frac{\delta_R F}{\delta\eta_i}\frac{\delta_L G}{\delta\eta^{i\dagger}} + \\ & - \frac{\delta_R F}{\delta\eta^{i\dagger}}\frac{\delta_L G}{\delta\eta_i} + \frac{\delta_R F}{\delta X^i}\frac{\delta_L G}{\delta X_i^{\dagger}} - \frac{\delta_R F}{\delta X_i^{\dagger}}\frac{\delta_L G}{\delta X^i} \Bigg]\end{aligned} \tag{4.32}$$

4.2.3 BV action and BV quantum master equation

The action of the BV Poisson Sigma model is formally

$$S\left[\widetilde{X}^i, \widetilde{\eta}_i\right] = \int_{T[1]\Sigma} \mu \left(\widetilde{\eta}_i d\widetilde{X}^i + \frac{1}{2}\alpha^{ij}\widetilde{\eta}_i\widetilde{\eta}_j \right), \tag{4.33}$$

where $\widetilde{X}^i$ and $\widetilde{\eta}_i$ are the superfields introduced in (4.25) and (4.26).

Classical BV master equation

We can demonstrate the classical BV equation, i.e.

$$\{S_{BV}, S_{BV}\}_{BV} = 0 \tag{4.34}$$

We can write action (4.33) in the following form

$$S_{BV} = S_0 + S_1, \tag{4.35}$$

where S_0 and S_1 are the two terms in equation (4.33). Indeed, we can split the classical master equation (4.34) in the following form

$$\{S_0, S_0\}_{BV} + 2\{S_0, S_1\}_{BV} + \{S_1, S_1\}_{BV} = 0 \tag{4.36}$$

Proof of relation (4.34)
In order to study the terms in (4.36) we need to calculate the following directional derivative. For the superfield $\widetilde{X}^i$, we have

$$\left.\frac{d}{dt} S_{BV}(\widetilde{X}^i + tb^i)\right|_{t=0} \tag{4.37}$$

Substituting (4.33) in (4.37) and we have

$$\begin{aligned}&\left.\frac{d}{dt}\int_{T[1]\Sigma} \mu \left(\widetilde{\eta}_i d\left(\widetilde{X}^i + tb^i\right) + \frac{1}{2}\alpha^{ij}\widetilde{\eta}_i\widetilde{\eta}_j \right)\right|_{t=0} = \\ &= \int_{T[1]\Sigma} \mu\ b^i \left(d\widetilde{\eta}_i + \frac{1}{2}\partial_i\alpha^{jk}\widetilde{\eta}_j\widetilde{\eta}_k \right)\end{aligned}$$

Hence

$$\frac{\delta_R S(\widetilde{X}^i + tb^i)}{\delta \widetilde{X}^i} = \frac{\delta_L S(\widetilde{X}^i + tb^i)}{\delta \widetilde{X}^i} = d\widetilde{\eta}_i + \frac{1}{2}\partial_i \alpha^{jk}\widetilde{\eta}_j\widetilde{\eta}_k \tag{4.38}$$

The directional derivative for the superfield $\widetilde{\eta}_i$ corresponds to

$$\frac{d}{dt} S_{BV}(\widetilde{\eta}_i + t\xi_i)\Bigg|_{t=0} \tag{4.39}$$

Substituting (4.33) in (4.39) and we have

$$\begin{aligned}
&\frac{d}{dt}\int_{T[1]\Sigma} \mu \left((\widetilde{\eta}_i + t\xi_i) d\widetilde{X}^i + \frac{1}{2}\alpha^{ij}(\widetilde{\eta}_i + t\xi_i)(\widetilde{\eta}_j + t\xi_j)\right)\Bigg|_{t=0} = \\
&= \int_{T[1]\Sigma} \mu \ \xi_i (d\widetilde{X}^i + \alpha^{ij}\widetilde{\eta}_i)
\end{aligned}$$

Hence

$$\frac{\delta_R S(\widetilde{\eta}_i + t\xi_i)}{\delta \widetilde{\eta}_i} = \frac{\delta_L S(\widetilde{\eta}_i + t\xi_i)}{\delta \widetilde{\eta}_i} = d\widetilde{X}^i + \alpha^{ij}\widetilde{\eta}_j \tag{4.40}$$

We study the first term in (4.36)

$$\{S_0, S_0\}_{BV} = 2\int_{T[1]\Sigma} \mu \ d\left(\widetilde{\eta}_i d\widetilde{X}^i\right) = 0$$

For the second term we have

$$\begin{aligned}
2\{S_0, S_1\} &= 2\int_{T[1]\Sigma} \mu \left(\frac{\delta_R S_0}{\delta \widetilde{X}^i}\frac{\delta_L S_1}{\delta \widetilde{\eta}_i} - \frac{\delta_R S_0}{\delta \widetilde{\eta}_i}\frac{\delta_L S_1}{\delta \widetilde{X}^i}\right) = \\
&= 2\int_{T[1]\Sigma} \mu \left(d\widetilde{\eta}_k \alpha^{kj}\widetilde{\eta}_j - \frac{1}{2} d\widetilde{X}^k \partial_k \alpha^{ij}\widetilde{\eta}_i\widetilde{\eta}_j \right) = \\
&= \int_{T[1]\Sigma} \mu \ d\left(\widetilde{\eta}_i \alpha^{ij}\widetilde{\eta}_j\right) + d\alpha^{ij}\widetilde{\eta}_i\widetilde{\eta}_j - d\widetilde{X}^k\partial_k\alpha^{ij}\widetilde{\eta}_i\widetilde{\eta}_j = 0
\end{aligned}$$

For the last term we have

$$\begin{aligned}
\{S_1, S_1\}_{BV} &= 2\int_{T[1]\Sigma} \mu \left(\alpha^{il}\partial_l\alpha^{jk}\widetilde{\eta}_i\widetilde{\eta}_j\widetilde{\eta}_k\right) = \\
&= \frac{2}{3}\int_{T[1]\Sigma} \mu \Big(\alpha^{il}\partial_l\alpha^{jk}\widetilde{\eta}_i\widetilde{\eta}_j\widetilde{\eta}_k + \alpha^{jl}\partial_l\alpha^{ki}\widetilde{\eta}_j\widetilde{\eta}_k\widetilde{\eta}_l + \alpha^{il}\partial_l\alpha^{jk}\widetilde{\eta}_k\widetilde{\eta}_i\widetilde{\eta}_j\Big) = \\
&= \frac{2}{3}\int_{T[1]\Sigma} \mu \Big(\alpha^{il}\partial_l\alpha^{ik} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij}\Big)\widetilde{\eta}_i\widetilde{\eta}_j\widetilde{\eta}_k = 0
\end{aligned}$$

where the last expression vanishes thanks to (4.1). Indeed, (4.36) holds true. □

Quantum BV master equation

From the theory of the BV formalism we know that the action (4.33) satisfies the quantum master equation, namely

$$\{S_{BV}, S_{BV}\}_{BV} - 2i\hbar\Delta_{BV}S_{BV} = 0 \tag{4.41}$$

Since the action (4.33) is closed; i.e. it obeys to relation (4.34), we write the previous equation as follows

$$2i\hbar\Delta_{BV}S_{BV} = \Delta_{BV}S_{BV} = 0 \tag{4.42}$$

where Δ_{BV} is the BV laplacian whose has the following form

$$\Delta_{BV} = \int_{T[1]\Sigma} \mu \left(\frac{\delta_L}{\delta \widetilde{X}^i} \frac{\delta_L}{\delta \widetilde{\eta}_i} \right) \tag{4.43}$$

For the Proof of relation (4.42), we remind to the next section where we perform this calculation using the action in field components.

4.2.4 BV Action in field components

We can express the BV action (4.33) in field components using the decomposition of the superfields (4.25) and (4.26). The final result is the following

$$\begin{aligned} S_{BV} = \int_\Sigma \Big[& -\eta^{i\dagger} d\beta_i + \eta_i dX^i + \alpha^{ij}(X) X_i^\dagger \beta_j + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j + \\ & -\frac{1}{2}\beta^{k\dagger}\partial_k\alpha^{ij}(X)\beta_i\beta_j - \eta^{k\dagger}\partial_k\alpha^{ij}(X)\eta_i\beta_j - \frac{1}{4}\eta^{k\dagger}\eta^{l\dagger}\partial_k\partial_l\alpha^{ij}(X)\beta_i\beta_j \Big] \end{aligned} \tag{4.44}$$

Proof of relation (4.44)
Using (4.25) and (4.26) in (4.33) we have

$$\begin{aligned} & \int_\Sigma \Big[\left(\beta_i + \eta_i + X_i^\dagger\right) d\left(X^i + \eta^{i\dagger} - \beta^{i\dagger}\right) + \\ & + \frac{1}{2}\Big(\alpha^{ij}(X) + \partial_k\alpha^{ij}(X)\left(\eta^{k\dagger} - \beta^{k\dagger}\right) + \partial_k\partial_l\alpha^{ij}(X)\eta^{k\dagger}\eta^{l\dagger}\Big) \\ & \left(\beta_i + \eta_i + X_i^\dagger\right)\left(\beta_j + \eta_j + X_j^\dagger\right) \Big] = \\ & = \int_\Sigma \Big(\beta_i d\eta^{i\dagger} + \eta_i dX^i + \alpha^{ij}(X)\beta_i X_i^\dagger + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j + \\ & + \partial_k\alpha^{ij}(X)\eta^{k\dagger}\eta_i\beta_j - \frac{1}{2}\beta^{k\dagger}\partial_k\alpha^{ij}(X)\beta_i\beta_j - \frac{1}{4}\eta^{k\dagger}\eta^{l\dagger}\partial_k\partial_l\alpha^{ij}(X)\beta_i\beta_j \Big) \end{aligned}$$

then (4.44) holds true. □

We introduce the BV field variations, namely

$$\delta_{BV}(\cdot) = \{S_{BV}, \cdot\}_{BV}, \tag{4.45}$$

where $\{\cdot,\cdot\}$ is the BV bracket in field components introduced in (4.32).
The BV field variations for the fields are

$$\delta_{BV}\beta_i = \frac{1}{2}\partial_k\alpha^{ij}\beta_i\beta_j \tag{4.46}$$

$$\begin{aligned}\delta_{BV}\eta_i = -d\beta_i - \partial_k\alpha^{ij}(X)\eta_i\beta_j + \\ - \frac{1}{2}\eta^{j\dagger}\partial_i\partial_j\alpha^{mn}\beta_m\beta_n\end{aligned} \tag{4.47}$$

$$\delta_{BV}X^i = \alpha^{ij}(X)\beta_j \tag{4.48}$$

while for the antifields we have

$$\begin{aligned}\delta_{BV}\beta^{i\dagger} =& - d\eta^{i\dagger} - \alpha^{ij}(X)X^\dagger_j + \partial_k\alpha^{ij}(X)\beta^{k\dagger}\beta_j + \\ &+ \partial_k\alpha^{ij}(X)\eta^{k\dagger}\eta_j + \frac{1}{2}\partial_j\partial_k\alpha^{il}\eta^{j\dagger}\eta^{k\dagger}\beta_l\end{aligned} \tag{4.49}$$

$$\delta_{BV}\eta^{i\dagger} = -dX^i - \alpha^{ij}(X)\eta_j - \partial_k\alpha^{ij}(X)\eta^{k\dagger}\beta_j \tag{4.50}$$

$$\begin{aligned}\delta_{BV}X^\dagger_i =& \, d\eta_i + \partial_i\alpha^{mn}(X)X^\dagger_m\beta_n + \frac{1}{2}\partial_i\alpha^{mn}\eta_m\eta_n + \\ &- \frac{1}{2}\partial_i\partial_j\alpha^{mn}(X)\beta^{j\dagger}\beta_m\beta_n - \partial_i\partial_k\alpha^{mn}(X)\eta^{k\dagger}\eta_m\beta_n + \\ &- \frac{1}{4}\partial_i\partial_j\partial_k\alpha^{mn}(X)\eta^{j\dagger}\eta^{k\dagger}\beta_m\beta_n\end{aligned} \tag{4.51}$$

BV Quantum Master Equation

Using the expression for the BV action in field components (4.44) we can verify the quantum master equation, namely

$$\Delta_{BV}S_{BV} = 0 \tag{4.52}$$

introduced yet in section 4.2. In this case the BV laplacian in field components reads as follows

$$\Delta_{BV} = \int_\Sigma \left(\frac{\delta_L}{\delta X^i}\frac{\delta_L}{\delta X^\dagger_i} - \frac{\delta_L}{\delta\beta_i}\frac{\delta_L}{\delta\beta^{i\dagger}} + \frac{\delta_L}{\delta\eta_i}\frac{\delta_L}{\delta\eta^{i\dagger}} \right) \tag{4.53}$$

Proof of relation (4.52)
Using the definition of the laplacian in field components and the expression of the action in field components (4.44), we have

$$\begin{aligned}\Delta_{BV}\Bigg(\int_\Sigma \eta_i dX^i + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j - \frac{1}{2}\beta^{k\dagger}\partial_k\alpha^{ij}\beta_i\beta_j + \alpha^{ij}X^\dagger_i\beta_j + \\ - \eta^{i\dagger}(d\beta_i + \partial_i\alpha^{jk}(X)\eta_j\beta_k) - \frac{1}{4}\eta^{i\dagger}\eta^{j\dagger}\partial_i\partial_j\alpha^{mn}(X)\beta_m\beta_n \Bigg)\end{aligned}$$

Considering only the terms whose contribute to the laplacian we obtain

$$\begin{aligned}&\Delta_{BV}\int_\Sigma \left(-\frac{1}{2}\beta^{k\dagger}\partial_k\alpha^{ij}(X)\beta_i\beta_j + \alpha^{ij}(X)X^\dagger_i\beta_j - \eta^{i\dagger}\partial_i\alpha^{jk}(X)\eta_j\beta_k \right) = \\ &= \delta(0)\int_\Sigma (1+1-2)\partial_i\alpha^{ij}(X)\beta_j = 0\end{aligned}$$

We have regularized the Dirac Delta function evaluated in 0 imposing a cut-off. In an adequate regularization scheme the previous result is supposed to be valid before removing the regularization or rather when $\delta(0)$ is infinite. □

4.2.5 Gauge fixing for the BV Poisson Sigma Model

In this section we study the gauge fixing procedure in order to quantize the model. First of all we introduce trivial pairs of fields/antifields.

$$\left(\overset{\sim i}{c}\right) \in \underline{\Gamma}(\Sigma, X^*T[-1]M) \qquad \left(\overset{\sim\dagger}{c}_i\right) \in \underline{\Gamma}(\Sigma, \Lambda^2 T^*\Sigma \otimes X^*T^*M) \tag{4.54}$$

$$\left(\overset{\sim i}{\lambda}\right) \in \underline{\Gamma}(\Sigma, X^*TM) \qquad \left(\overset{\sim\dagger}{\lambda}_i\right) \in \underline{\Gamma}(\Sigma, \Lambda^2 T^*\Sigma \otimes X^*T^*[-1]M) \tag{4.55}$$

The BV auxiliary symplectic form reads as follows

$$\Omega_{BVaux} = \int_\Sigma \left(\delta \overset{\sim\dagger}{c}_i \delta \overset{\sim i}{c} + \delta \overset{\sim\dagger}{\lambda}_i \delta \overset{\sim i}{\lambda}\right) \tag{4.56}$$

The related BV bracket has the following form

$$\{F, G\}_{BVaux} = \int_\Sigma \left[\frac{\delta_R F}{\delta \overset{\sim i}{c}} \frac{\delta_L G}{\delta \overset{\sim\dagger}{c}_i} - \frac{\delta_R F}{\delta \overset{\sim\dagger}{c}_i} \frac{\delta_L G}{\delta \overset{\sim i}{c}} + \frac{\delta_R F}{\delta \overset{\sim i}{\lambda}} \frac{\delta_L G}{\delta \overset{\sim\dagger}{\lambda}_i} - \frac{\delta_R F}{\delta \overset{\sim\dagger}{\lambda}_i} \frac{\delta_L G}{\delta \overset{\sim i}{\lambda}}\right] \tag{4.57}$$

Auxiliary BV action

We can introduce the Auxiliary BV action as follows

$$S_{BVaux} = -\int_\Sigma \overset{\sim\dagger}{c}_i \overset{\sim i}{\lambda} \tag{4.58}$$

From a direct inspection the following relation holds true

$$\{S_{BVaux}, S_{BVaux}\}_{BVaux} = 0 \tag{4.59}$$

(4.59) corresponds to the classical master equation for the auxiliary BV action (4.58).

We can introduce the auxiliary BV variations as follows

$$\delta_{BVaux}(\cdot) = \{S_{BVaux}, \cdot\}_{BVaux} \tag{4.60}$$

The auxiliary BV variations for the fields and antifields introduced in (4.54) and (4.55) are

$$\delta_{BVaux} \overset{\sim i}{c} = -\overset{\sim i}{\lambda} \tag{4.61}$$

$$\delta_{BVaux} \overset{\sim\dagger}{c}_i = 0 \tag{4.62}$$

$$\delta_{BVaux} \overset{\sim i}{\lambda} = 0 \tag{4.63}$$

$$\delta_{BVaux} \overset{\sim\dagger}{\lambda}_i = -\overset{\sim\dagger}{c}_i \tag{4.64}$$

As expected we have nilpotence property for the auxiliary BV variations, namely

$$\delta^2_{BVaux}(\cdot) = 0 \tag{4.65}$$

Proof of relation (4.65)

$$\delta^2_{BVaux}\left(\tilde{c}^i, \tilde{c}^\dagger_i, \overset{\sim}{\lambda}^i, \overset{\sim}{\lambda}^\dagger_i\right) = \delta_{BVaux}\left(-\overset{\sim}{\lambda}^\dagger_i, 0, 0, -\tilde{c}^\dagger_i\right) = 0$$

then (4.65) holds true. $\square$

The auxiliary BV action (4.58) is invariant under the auxiliary BV variations, ie.

$$\delta_{BVaux} S_{BVaux} = 0 \tag{4.66}$$

Proof of relation (4.66)

$$\delta_{BVaux}\left(-\int_\Sigma \tilde{c}^\dagger_i \overset{\sim}{\lambda}^i\right) = -\int_\Sigma \left(\delta_{BVaux}\tilde{c}^\dagger_i \overset{\sim}{\lambda}^i + \tilde{c}^\dagger_i \delta_{BVaux}\overset{\sim}{\lambda}^i\right) = 0$$

then (4.66) holds true. $\square$

The Gauge Fermion

The gauge fermion for the BV Poisson Sigma Model is the formally

$$\Psi = -\int_\Sigma d\tilde{c}^i \star \eta_i \tag{4.67}$$

where $\star$, as usual, is the Hodge star operator
Using (4.67) we can define a lagrangian submanifold $\mathscr{L}$ in the field space as follows

$$\varphi^\dagger_A = \frac{\delta_L \Psi}{\delta \varphi^A}, \tag{4.68}$$

one finds

$$\tilde{c}^\dagger_i = -d \star \eta_i \tag{4.69}$$

$$\overset{\sim}{\lambda}^\dagger_i = 0 \tag{4.70}$$

$$\eta^{i\dagger} = -\star d\tilde{c}^i \tag{4.71}$$

$$X^\dagger_i = 0 \tag{4.72}$$

$$\beta^{i\dagger} = 0 \tag{4.73}$$

Thanks to the gauge fermion (4.67) we can define the gauge fixed action, namely

$$I = (S_{BV} + S_{BVaux})|_{\mathscr{L}} \tag{4.74}$$

Substituting in (4.33) and (4.58) in (4.74) we obtain an explicit expression for the gauge fixed action, i.e.

$$\begin{aligned}
&\left(\int_\Sigma \eta_i dX^i + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j - \frac{1}{2}\beta^{k\dagger}\partial_k\alpha^{ij}\beta_i\beta_j + \alpha^{ij}X^\dagger_i\beta_j + \right.\\
&\left.\left. - \eta^{i\dagger}(d\beta_i + \partial_i\alpha^{jk}(X)\eta_j\beta_k) - \frac{1}{4}\eta^{i\dagger}\eta^{j\dagger}\partial_i\partial_j\alpha^{mn}(X)\beta_m\beta_n - \tilde{c}^\dagger_i\overset{\sim}{\lambda}^i\right)\right|_{\mathscr{L}} = \\
&= \int_\Sigma \eta_i dX^i + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j + \star d\tilde{c}^i(d\beta_i + \partial_i\alpha^{ij}(X)\eta_j\beta_k) + \\
&- \frac{1}{4}\star d\tilde{c}^i \star d\tilde{c}^j\partial_i\partial_j\alpha^{mn}(X)\beta_m\beta_n + (d\star\eta_i)\overset{\sim}{\lambda}^i.
\end{aligned}$$

The gauge fermion (4.67) is not a global well defined functional. The ghost $\tilde{c}^{i}$ does not transform in a covariant way, namely

$$d\tilde{c}^{i'} = d\tilde{c}^{j}\frac{\partial x^{i'}}{\partial x^{j}} - \tilde{c}^{j} dx^{k}\frac{\partial^2 x^{i'}}{\partial x^{k}\partial x^{j}} \tag{4.75}$$

A possible solution for this problem is considering a connection in the target space. In this case we can provide a new version of the gauge fermion, i.e.

$$\Psi' = -\int_{\Sigma} \tilde{c}^{i}\nabla \star \eta_i = -\int_{\Sigma} \nabla\tilde{c}^{i} \star \eta_i = -\int_{\Sigma}\left(d\tilde{c}^{i} + dX^{j}\Gamma^{i}_{jk}\tilde{c}^{k}\right)\star\eta_i, \tag{4.76}$$

where Γ^{i}_{jk} is the connection introduced in the target space.
Using (4.76) we can select a new lagrangian submanifold $\mathscr{L}'$ in the field space as follows

$$\varphi^{\dagger}_{A} = \frac{\delta_L \Psi'}{\delta\varphi^{A}} \tag{4.77}$$

one finds

$$\tilde{c}^{\dagger}_{i} = -\nabla\star\eta_i \tag{4.78}$$

$$\overset{\sim}{\lambda}^{\dagger}_{i} = 0 \tag{4.79}$$

$$\eta^{i\dagger} = -\star\nabla\tilde{c}^{i} \tag{4.80}$$

$$X^{\dagger}_{i} = 0 \tag{4.81}$$

$$\beta^{i\dagger} = 0 \tag{4.82}$$

We can formulate the new gauge fixed action (4.74) using the lagrangian submanifold $\mathscr{L}'$, then we obtain

$$\begin{aligned} &= \int_{\Sigma}\eta_i dX^{i} + \frac{1}{2}\alpha^{ij}(X)\eta_i\eta_j + \star\nabla\tilde{c}^{i}(d\beta_i + \partial_i\alpha^{ij}(X)\eta_j\beta_k) + \\ &+ \frac{1}{4}\nabla\tilde{c}^{i}\nabla\tilde{c}^{j}\partial_i\partial_j\alpha^{mn}(X)\beta_m\beta_n + (\nabla\star\eta_i)\overset{\sim}{\lambda}^{i}. \end{aligned} \tag{4.83}$$

The gauge fixed action (4.83) has a well defined kinetic terms.

Outlook and Open problems

In this thesis, it has been presented three models in a BV perspective: 3d Chern-Simons theory, BF theory and Poisson Sigma Model. The Batalin-Vilkovisky quantization technique can be applied to many other different topological quantum field theories. In this last section, the Courant sigma model and the related gauge fixing problem are outlined to suggest possible future lines of research.

Courant Sigma Model

First of all we introduce Courant algebroids, then we illustrate the action of the Courant sigma model.

Courant algebroid

Let $V \longrightarrow X$ be a metric vector bundle. Therefore, there is a degree 2 symplectic $N-$manifold L related to V as follows. Given the 2-shifted cotangent bundle $T^*[2]V[1]$ of the 1−shifted bundle $V[1]$, then $T^*[2]V[1]$ is a degree 2 symplectic $N-$manifold which can be locally described by degree 0 base coordinates x^i, degree 1 fiber coordinates ξ^a of $V[1]$ and by related cotangent degree 2 base coordinates p_i and degree 1 fiber coordinates η_a. Furthermore $T^*[2]V[1]$ is endowed with the following canonical degree 2 symplectic 2-form

$$\omega_0 = dp_i dx^i + d\eta_a d\xi^a \tag{5.1}$$

We consider, for simplicity, a local trivialization of V such that the coefficients g_{ab} of the metric of V are constant. Thus the following covariant constraint

$$\eta_a = \frac{1}{2} g_{ab}\xi^b \tag{5.2}$$

defines a submanifold M of $T^*[2]V[1]$. M is a degree 2 symplectic $N-$manifold. M is coordinatized by the degree 0,1,2 coordinates x^i, ξ^a, p_i and is equipped with the following degree 2 symplectic 2-form

$$\omega = dp_i dx^i + \frac{1}{2} d\xi^a g_{ab} d\xi^b \tag{5.3}$$

yielded by the pull-back of (5.1) by the embedding $M \longrightarrow T^*[2]V[1]$. Conversely, it can be demonstrated that every degree 2 symplectic $N-$manifold M arises from a

metric vector bundle V by the above construction.
Thanks to the fact that the constraint (5.2) defining the embedding into $T^*[2]V[1]$ is linear, M can be identified with the 1-shift $L[1]$ of a graded vector bundle L over N. This is not a canonical result, but it depends on an arbitrary choice of a metric connection of V.

The metric vector bundle V is a Courant algebroid if the graded vector bundle L has the so called L_∞-algebroid structure with the homological vector field Q_L of $L[1]$ hamiltonian with the respect to the Poisson bracket associated to the symplectic form (5.3). In that case Q_L has the following form

$$\begin{aligned} Q_L = \rho^i{}_a(x)\xi^a p_i\partial_{x_i} + \left(-\partial_{x_i}\rho^j{}_a(x)\xi^a p_j + \frac{1}{6}\partial_{x_i} f_{abc}\xi^a\xi^b\xi^c\right)\partial_p{}^i + \\ + g^{ad}\left(-p^i{}_d(x)p_i + \frac{1}{2}f_{dbc}(x)\xi^b\xi^c\right)\partial_{\xi_a} \end{aligned} \tag{5.4}$$

for a certain functions $\rho^i{}_a$ and f_{abc}. Therefore, since Q_L is hamiltonian, $Q_L = \{S, \cdot\}$ for some degree 3 function S on $L[1]$ can be locally expressed as follows

$$S = -\rho^i{}_a\xi^a p_i + \frac{1}{6}f_{abc}\xi^a\xi^b\xi^c \tag{5.5}$$

leading to (5.4).
The nilpotence ofQ_L is equivalent to the following equation

$$\{S, S\} = 0 \tag{5.6}$$

the structure functions $\rho^i{}_a$ and f_{abc} define the Courant anchor and bracket of V respectively. Equation (5.6) implies a set of relations that characterized V as a Courant algebroid and that the structure functions $\rho^i{}_a$ and f_{abc} must obey.
For a Courant algebroid V, $L[1]$ is therefore a degree 2 $PQ-$manifold. In [27] there is the proof that every degree 2 $PQ-$manifold stems a Courant algebroid V thanks to the above construction.

Courant sigma model

The Courant sigma model, introduced by Ikeda in [19], and treated by Roytenberg in [27] via AKSZ formalism consists in a topological quantum field theory of Schwarz type. The classical action of this model reads as follows

$$S[A^a, B_i, X^i] = \int_M \left[\frac{1}{2}A^a g_{ab} dA^b + B_i dX^i - \rho^i{}_a(X)A^a B_i + \frac{1}{6}f_{abc}A^a A^b A^c\right] \tag{5.7}$$

where $\rho^i{}_a(x)$ and f_{abc} are the structure constants of the Courant algebroid previously introduced and X^i, A^a and B_i are forms of degree 0,1 and 2 respectively. We can formulate this theory in a BV perspective. Nowadays for the BV formulation of this model there is no treatment of the gauge fixing procedure. This remains an open problem left for future work.

Appendix A

Supergeometry/Graded geometry

In this chapter we introduce some elements of supergeometry and graded geometry (an extension of the ordinary geometry) and related spaces (called superspace and graded space respectively). We also present many examples of graded manifolds and the related theory of integration.

A.1 The fundamental idea behind supergeometry

Supergeometry is an extension of the ordinary geometry with anticommuting coordinates in addition to the usual even ones.
We consider a smooth manifold M and the algebra of smooth functions $C^\infty(M)$ over M. $C^\infty(M)$ is a commutative ring. The functions which vanish on a given subspace of M form an ideal of this ring. The maximal ideals would correspond to the points of M. In a supergeometric context we replace the usual ring of functions, with supercommutative ring and the supermanifold arises

A.2 $\mathbb{Z}_2$-linear algebra

A.2.1 superspace

The rings of functions of supermanifolds is $\mathbb{Z}_2$-graded. It is necessary to study $\mathbb{Z}_2$-graded linear algebra. We study $\mathbb{Z}_2$-graded vector space V over $\mathbb{R}$(or $\mathbb{C}$) as a direct decomposition:

$$V = V_0 \oplus V_1$$

where V_0 is called even and V_1 is called odd.
Define now the space of homogeneous elements as follows:

$$V_H = V_0 \cup \ V_1 \tag{A.1}$$

Considering an element $f \in V_H$, we can define the parity for f.

Definition 1.1 we denote the parity (or degree) of an element $f \in V_H$ as $|f|$.
It's defined as follows:

-if $f \in V_0$, so f is even, $|f| = 0$.
-if $f \in V_1$, so f is odd, $|f| = 1$.

A generic element $f \in V$ may doesn't have a definite degree.

In what follows, unless explicity stated otherwise, we assume that all elements of V belong to either V_0 or V_1 and have a definite degree.

Definition 1.2 Let V, W superspace. A superspace morphism is a linear map

$$T : V \longrightarrow W \tag{A.2}$$

such that:

$$TV_0 \subset W_0 \qquad TV_1 \subset W_1 \tag{A.3}$$

Let $Hom(V, W)$ denote the vector space of morphisms $V \longrightarrow W$.

Definition 1.3 Let V, W superspaces. A inner morphism of superspaces is a linear map

$$T : V \longrightarrow W \tag{A.4}$$

such that

$$TV_0 \subset W_0 \qquad TV_1 \subset W_1 \qquad (Inner\ even) \tag{A.5}$$

or

$$TV_0 \subset W_1 \qquad TV_1 \subset W_0 \qquad (Inner\ odd) \tag{A.6}$$

We denote the inner morphism of superspaces by $\underline{Hom}(V, W)$.

If dim $V_0 = d_0$ and dim $V_1 = d_1$, then we adopt the following notation $V^{d_0|d_1}$ and the combination (d_0, d_1) is called the superdimension of V.

A.2.2 Superalgebras

Definition 1.4 A commutative superalgebra A is a superspace equipped with a distributive and associative multiplication

$$A \times A \longrightarrow A \tag{A.7}$$

such that:
1) For homogeneous $a, b \in A$

$$a\ b = (-1)^{|a||b|} b\ a \tag{A.8}$$

2)For homogeneous $a, b \in A$, $a \cdot b$ is homogeneous with

$$|ab| = |a| + |b| \qquad (mod\ 2) \tag{A.9}$$

Definition 1.5 A superderivation D of a commutative superalgebra A is an inner endomorphism of the superspace A such that

$$D(ab) = Da\ b + (-1)^{|D||a|} a\ Db \tag{A.10}$$

We discuss now an important example of supercommutative algebra, the exterior algebra.

Example 1.1 consider purely odd superspace $R^{0|m}$. Let us pick up the basis ξ^i,i=1,2,...,m and define the multiplication between the basis elements satisfying $\xi^i\xi^j = -\xi^j\xi^i$. The functions $C^\infty(\mathbb{R}^{0|m})$ on $\mathbb{R}^{0|m}$ are given by the following expression

$$f(\xi^1, \xi^2, ..., \xi^m) = \sum_{l=0}^{m} \frac{1}{l!} f_{i_1 i_2 ... i_m} \xi^{i_1} \xi^{i_2} ... \xi^{i_l} \tag{A.11}$$

and they correspond to the elements of the exterior algebra $\wedge^\bullet(\mathbb{R}^m)^*$. The exterior algebra

$$\wedge^\bullet (\mathbb{R}^m)^* = (\wedge^{even}(\mathbb{R}^m)^*) \bigoplus (\wedge^{odd}(\mathbb{R}^m)^*) \tag{A.12}$$

is a supervector space with the supercommutative multiplications given by wedge product. In this case the wedge product corresponds to the function multiplication in $C^\infty(\mathbb{R}^{0|m})$.

A.2.3 Lie superalgebras

Definition 1.6 A Lie superalgebra $\mathfrak{g}$ is a superspace equipped with Lie superbrackets, that is a bilinear degree preserving map

$$[-, -] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \tag{A.13}$$

such that

$$[a, b] + (-1)^{|a||b|} [b, a] \tag{A.14}$$

and

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|} [b, [a, c]] = 0 \tag{A.15}$$

We present some examples of commutative Lie superalgebras.

Example 1.2 Let A be a commutative superalgebra.
Define $[-, -]; A \times A \longrightarrow A$ by

$$[a, b] = ab - (-1)^{|a||b|} ba \tag{A.16}$$

Then, $(A, [-, -])$ is a Lie superalgebra $\mathfrak{g}_A$.
If $A = \underline{End}(V)$, then this Lie superalgebra is called $\mathfrak{gl}(V)$

Example 1.3 Let A a commutative superalgebra with derivation $Der(A)$. $Der(A)$ is a superspace. We define the following Lie bracket:

$$[-, -] : Der(A) \times Der(A) \longrightarrow Der(A) \tag{A.17}$$

by

$$[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1 \tag{A.18}$$

then $Der(A)$ is a Lie superalgebra.

A.2.4 Supermanifolds

Now we can construct more complicated examples of supercommutative algebras. Consider, for example, the real superspace $\mathbb{R}^{n|m}$ and define the space of functions on it as follows:

$$C^\infty(\mathbb{R}^{n|m}) \equiv C^\infty(\mathbb{R}^n) \bigotimes \wedge^\bullet(\mathbb{R}^m)^* \tag{A.19}$$

if we pick up an open set U_0 in $\mathbb{R}^n$, then we can associate to U_0 the supercommutative algebras as follows:

$$U_0 \longrightarrow C^\infty(U_0) \bigotimes \wedge^\bullet(\mathbb{R}^m)^* \tag{A.20}$$

This supercommutative algebra can be thought as the algebra of functions on the superdomain $U^{n|m} \subset \mathbb{R}^{n|m}$, $C^\infty(U_0) \bigotimes \wedge^\bullet(\mathbb{R}^m)^*$.
The superdomain can be characterized in terms of standard even coordinates x^μ,$\mu = 1, 2, ..., n$ for U_0 and the odd coordinates ξ^i, $i = 1, 2, ..., m$.

Now we can introduce the definition of smooth supermanifold:

Definition 1.7 A smooth supermanifold M of a dimension (n,m) is a smnooth manifold M with a sheaf of supercommutative algebras, typically denoted O_M or C^∞_M, that is locally isomorphic to $C^\infty(U_0) \otimes \wedge^\bullet(\mathbb{R}^m)^*$, where U_0 is an open subset of $\mathbb{R}^n$.
Essentially the theory of supermanifolds mimics the standard smooth manifolds.
We can present some important examples of supermanifolds:

Example 1.4 Given a manifold M, the parity reversed tangent bundle ΠTM is the usual tangent bundle in which the fiber is assumed to be odd (of Grassman degree 1). Under a change of local coordinates, x and ξ transform in the usual way.

$$\tilde{x}^\mu = \tilde{x}^\mu(x) \qquad \tilde{\xi}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\xi^\nu \tag{A.21}$$

where x's are local coordinates on M and ξ's are odd. The functions on this supermanifold have the following expansion:

$$f(x, \xi) = \sum_{p=0}^{dimM} \frac{1}{p!} f_{\mu_1\mu_2...\mu_p}(x)\xi^{\mu_1}\xi^{\mu_2}...\xi^{\mu_p} \tag{A.22}$$

and thus they are naturally identified with the differential forms on M, $C^\infty(\Pi TM) = \Omega^{even/odd}(M)$.

Example 1.5 Given, as in the previous example, M a smooth manifold. The parity reversed cotangent bundle ΠT^*M is the usual cotangent bundle in which the fiber is assumed to be odd. Under a change o local coordinates x and ξ transform in the usual way.

$$\tilde{x}^\mu = \tilde{x}^\mu(x) \qquad \tilde{\xi}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\xi_\nu \tag{A.23}$$

In this example x's are local coordinates on M and ξ's transform as ∂_μ.
The functions on ΠT^*M have the following expansion

$$f(x,\xi) = \sum_{p=0}^{dimM} \frac{1}{p!} f^{\mu_1\mu_2...\mu_p}(x)\xi_{\mu_1}\xi_{\mu_2}...\xi_{\mu_p} \tag{A.24}$$

and thus they are naturally identified with multivectors fields, $C^\infty(\Pi T^*M) = \Gamma(\wedge^\bullet TM)$.

Note that many notions and results from the standard differential geometry can be extended to supermanifold in a straightforward way.

A.3 Integration theory for supermanifolds

In this section we shall discuss integration on supermanifolds. There's a problem in this procedure related to the occurance of odd variables. A natural question arises about how to perform integration of odd variables. to solve this problem we must understand in which way we can treat them. Before this we have to recall some notions about integration theory for even variables and looking for indications about how to generalize to odd ones. We know from the ordinary calculus that the integral of a function $f(x)$ on an interval I can be performed splitting the interval in a very large number of small intervals I_i on which $f(x)$ is approximately constant, $f(x) = f_i$ and then sum up these values multiplied by the lenghts of the underlying intervals (1.18)

$$\int_I f(x)dx \simeq \sum_i f_i I_i \tag{A.25}$$

When we consider odd coordinates the above definition is useless because odd directions have no points and no intervals. According to the well known fundamental theorem of calculus, one has:

$$\int_a^b \frac{df(x)}{dx}dx = f(b) - f(a) \tag{A.26}$$

This important relation offers clues about the problem we are treating. In the case of a periodic function, it's defined on a circle (imposing $f(a) = f(b)$), that integral holds

$$\int_{S_1} \frac{df(x)}{dx}dx = 0 \tag{A.27}$$

It turn out that relation determines integral over the circle up to a multiplicative constant.

The same previous relation, but for odd variables is the following:

$$\int \frac{df(\xi)}{d\xi}d\xi \tag{A.28}$$

for a suitible differentiation respect to an odd variable, we discussed shortly. In a general approach we define a set of vector fields (first order differential operators).

$$D_a = v_a^i(x)\frac{\partial}{\partial x^i} \tag{A.29}$$

such that,given a manifold M

$$\int_M \mu D_a f = 0 \tag{A.30}$$

Where μ is some integration measure which nature made precise below.
Note that the differential operators (1.21) form a Lie algebra.

A.3.1 Berenzin integral

Felix Berenzin understood in which sense odd variables may be treated as the even ones.
Consider for example the space $\mathbb{R}^{1|1}$, coordinatized by (x, ξ). Since $\xi^2 = 0$, the most general function $f(x, \xi)$ has the form

$$f(x, \xi) = f_0(x) + \xi f_1(x) \tag{A.31}$$

The most natural derivative relations respect the odd and the even coordinate are:

$$even \qquad \frac{\partial}{\partial x} f(x, \xi) = \frac{df_0(x)}{dx} + \xi\frac{df_1(x)}{dx} \tag{A.32}$$

$$odd \qquad \frac{\partial}{\partial \xi} f(x, \xi) = f_1(x) \tag{A.33}$$

An important remark about relation (1.33). There two type of derivatives called right and left. The difference between them is the following:

$$Right \quad \frac{\partial_R}{\partial \xi} f(x, \xi) = \frac{\overrightarrow{\partial}}{\partial \xi} f(x, \xi) = f_1(x) \tag{A.34}$$

$$Left \quad \frac{\partial_L}{\partial \xi} f(x, \xi) = \frac{\overleftarrow{\partial}}{\partial \xi} f(x, \xi) = -f_1(x) \tag{A.35}$$

We can now integrate the previous relations, mimicking (1.27), so:

$$even \qquad \int dx \frac{\partial}{\partial x} f(x, \xi) = f(+\infty, 0) - f(-\infty, 0) \tag{A.36}$$

$$odd \qquad \int d\xi \frac{\partial}{\partial \xi} f(x, \xi) = 0 \tag{A.37}$$

In the integration theory we assume that $f(x, \xi)$ falls out rapidly to be integral in the usual sense.
A prescription for (1.37) holds in the following:

$$\int d\xi = \frac{\partial}{\partial \xi} \tag{A.38}$$

Since $\xi^2 = 0$ we have the following property:

$$\frac{\partial^2}{\partial \xi^2} f(x,\xi) = 0 \tag{A.39}$$

With this in hand, we can define a notion of integration on the space $\mathbb{R}^{1|1}$ of the function $f(x,\xi)$

$$\int dx d\xi f(x,\xi) = \int dx \frac{\partial}{\partial \xi} f(x,\xi) \tag{A.40}$$

We can generalize (1.40) on the space $\mathbb{R}^{n|p}$, which has coordinates $x^1 x^2 ... x^n, \xi^1 \xi^2 ... \xi^p$. Consider the following function:

$$f : \mathbb{R}^{n|p} \longrightarrow \mathbb{R} \tag{A.41}$$

the relative expansion is:

$$f(x,\xi) = \sum_{k=0}^{p} \frac{1}{k!} \xi^{i_1} ... \xi^{i_k} f_{i_1 ... i_k}(x^1 ... x^k) \tag{A.42}$$

integrating (1.42) and using the relation (1.38),we obtain

$$\int d^n x d^p \xi f(x,\xi) = \int d^n x \frac{\partial}{\partial \xi^p} ... \frac{\partial}{\partial \xi^1} f(x,\xi) = \tag{A.43}$$

$$= \int d^n x f_{i_1 ... i_p}(x^1 ... x^n)$$

Note that the integral selects the component $f_{i_1 ... i_p}$.

A.4 $\mathbb{Z}$-graded linear algebra

A.4.1 $\mathbb{Z}$-graded vector space

A $\mathbb{Z}$-graded vector space is a vector space V with a decompostion of the form labelled by integers:

$$V = \bigoplus_{i \in \mathbb{Z}} V_i \tag{A.44}$$

If $v \in V_i$, then we say that v is a homogeneous element of V a variable degree $|v| = i$. Any element of V can be decomposed in terms of homogeneous elements of a given degree. The morphism between graded vector spaces is defined as a linear map which preserves the grading.
The dual vector space of a $\mathbb{Z}$ graded is self-graded:

$$V^* = \bigoplus_{i \in \mathbb{Z}} (V^*)_i \tag{A.45}$$

with:

$$V^*{}_i = V_{-i}{}^* \tag{A.46}$$

A k shifted vector space V is the graded vector space $V[k]$ defined by:

$$V[k]_i = V_{k+i} \tag{A.47}$$

A k shifted dual vector space V is the graded vector space $V[k]$ defined by:

$$V[k]^*{}_i = V^*[-k]_i \tag{A.48}$$

If the graded vector space V is equipped with the associative product which respects the grading, then we call V a graded algebra. If we consider a graded algebra V and consider homogeneous elements v and $\widetilde{v}$ therein, the following relation holds true:

$$v\widetilde{v} = (-1)^{|v||\widetilde{v}|}\widetilde{v}v \tag{A.49}$$

then we call V a graded commutative algebra.

A derivation D of a grade $|D|$ is an endomorphism of graded algebra V with $\mathbb{Z}$ satisfying relation (1.10) with $\mathbb{Z}_2$ grading rule replaced by $\mathbb{Z}$ grading.
In the following, we present two of the most important examples of graded algebras

Example 1.6 Let V be a graded vector space over $\mathbb{R}$ (or $\mathbb{C}$). We define the graded symmetric algebra $S(V)$ as the linear space spanned by polynomial functions on V

$$\sum_l f_{a_1 a_2 \ldots a_l} v^{a_1} v^{a_2} \ldots v^{a_l} \tag{A.50}$$

we use the relation

$$v^a v^b = (-1)^{|v^a||v^b|} v^b v^a \tag{A.51}$$

with v^a and v^b are homogeneous elements of degree $|v^a|$ and $|v^b|$ respectively. The functions on V are naturally graded and multipilcation of functions is graded commutative. Therefore the graded symmetric algebra $S(V)$ is a graded commutative algebra.

Example 1.7 Let V be a graded vector space over $\mathbb{R}$ (or $\mathbb{C}$). We define now the graded antisymmetric algebra $S(V)$ as the linear space spanned by polynomial functions on V

$$\sum_l f_{a_1 a_2 \ldots a_l} v^{a_1} v^{a_2} \ldots v^{a_l} \tag{A.52}$$

we use the relation

$$v^a v^b = -(-1)^{|v^a||v^b|} v^b v^a \tag{A.53}$$

with v^a and v^b are homogeneous elements of degree $|v^a|$ and $|v^b|$ respectively. The functions on V are naturally graded and multiplication of functions is graded commutative. Therefore the graded antisymmetric algebra $S(V)$ is a graded commutative algebra.

A.4.2 Graded manifolds

In a close analogy with the section 1.2.4, we can define a graded manifold as an ordinary manifold coordinatized by even and odd coordinates(with a definite degree) and we glue them by the degree preserving maps. This is an intuitive definition. Let introduce now the formal one:

Definition 1.8 A smooth graded manifold M is a smooth manifold M with a sheaf of graded commutative algebras, usually denoted by C^∞_M, which is locally isomorphic to $C^\infty(U_0) \otimes S(V)$, where U_0 is an open subset of $\mathbb{R}^n$ and V is a graded vector space.

To understand better the idea of graded manifold we can introduce the following explicit examples.

Example 1.8 Exists a graded version of tangent bundle introduced in the example 1.4. This is the 1-shifted tangent bundle $T[1]M$, whose fiber is parametrized by degree 1 coordinates. The base coordinates x has degree 0. We have the same coordinates transformations as shown in example 1.4. The space of functions $C^\infty(T[1]M) = \Omega^\bullet(M)$ is a graded commutative algebra with the same $\mathbb{Z}$-grading as the differential forms.

Example 1.9 In the same way we can introduce the graded version $T^*[-1]M$ of the odd cotangent bundle (Example 1.5). We assign grade 0 and -1 to ξ and x respectively. As in the previous example the gluing rules preserve the degrees. The space of functions $C^\infty(T^*[-1]M) = \Gamma(\wedge^\bullet TM)$ is isomorphic to algebra of differentiable grade integers, with the opposite grading degree of multivector field.

The previous examples are very important in the BV-formalism. We'll see this formalism in the following chapter.

Appendix B

BV bracket

The aim of this appendix is to present the BV bracket using superfields and field components and provide a proof for these formulas using the standard results of the graded symplectic geometry.

B.1 BV bracket

In this section we provide an expression for the BV bracket using superfields. To obtain this result we can consider the following BV symplectic form

$$\Omega_{BV} = \int_{T[1]M} \mu \left(\hat{\delta}\underline{\varphi}^{\dagger} \hat{\delta}\underline{\varphi} \right), \tag{B.1}$$

where μ is the standard measure of $T[1]M$, μ has $T[1]M$ degree $-m$. We denote with $\hat{\delta}$ the functional de Rahm operator. $\underline{\varphi}$ and $\underline{\varphi}^{\dagger}$ are two de Rham superfields of degree p and $m-1-p$ respectively.
Given a functional f, we have

$$\begin{aligned} \hat{\delta} f &= \int_{T[1]M} \mu \left(\hat{\delta}\underline{\varphi} \frac{\delta_L f}{\delta \underline{\varphi}} + \hat{\delta}\underline{\varphi}^{\dagger} \frac{\delta_L f}{\delta \underline{\varphi}^{\dagger}} \right) + \\ &= \int_{T[1]M} \mu \left(\frac{\delta_R f}{\delta \underline{\varphi}} \hat{\delta}\underline{\varphi} + \frac{\delta_R f}{\delta \underline{\varphi}^{\dagger}} \hat{\delta}\underline{\varphi}^{\dagger} \right) \end{aligned} \tag{B.2}$$

We consider now the inner contraction i_{X_f}, whose has the following form

$$i_{X_f} = \int_{T[1]M} \mu \left(X \frac{\delta_L}{\delta \hat{\delta}\underline{\varphi}^{\dagger}} + X^{\dagger} \frac{\delta_L}{\delta \hat{\delta}\underline{\varphi}} \right), \tag{B.3}$$

where X_f is the hamiltonian vector field, which has the following form

$$X_f = \int_{T[1]M} \mu \left(X \frac{\delta_L}{\delta \underline{\varphi}^{\dagger}} + X^{\dagger} \frac{\delta_L}{\delta \underline{\varphi}} \right) \tag{B.4}$$

We apply (B.3) to (B.1), then

$$
\begin{aligned}
i_{X_f}\Omega_{BV} &= i_{X_f}\int_{T[1]M}\mu\left(\hat{\delta}\underline{\varphi}^{\dagger}\hat{\delta}\underline{\varphi}\right) = (-1)^{-m(|f|+1)}\int_{T[1]M}\mu\ i_{X_f}\left(\hat{\delta}\underline{\varphi}^{\dagger}\hat{\delta}\underline{\varphi}\right) = \\
&= (-1)^{-m(|f|+1)}\int_{T[1]M}\mu\ i_{X_f}\hat{\delta}\underline{\varphi}^{\dagger}\hat{\delta}\underline{\varphi} - (-1)^{(|f|+1)(m-1-p)}\hat{\delta}\underline{\varphi}^{\dagger}i_{X_f}\hat{\delta}\underline{\varphi} = \\
&= (-1)^{-m(|f|+1)}\int_{T[1]M}\mu\left((-1)^{m(|f|+1+p)}X\hat{\delta}\underline{\varphi} - (-1)^{(|f|+1)(m-1-p)+m(|f|+p+1)}\hat{\delta}\underline{\varphi}^{\dagger}X^{\dagger}\right) = \\
&= \int_{T[1]M}\mu\left((-1)^{mp}X\hat{\delta}\underline{\varphi} - (-1)^{(|f|+1)(m+p+1)+mp+(|f|+p+1)(m+p+1)}X^{\dagger}\hat{\delta}\underline{\varphi}^{\dagger}\right) = \\
&= \int_{T[1]M}\mu\left((-1)^{mp}X\hat{\delta}\underline{\varphi} - X^{\dagger}\hat{\delta}\underline{\varphi}^{\dagger}\right)
\end{aligned}
$$

We use now the well-known relation

$$
i_{X_f}\Omega_{BV} = \hat{\delta}f \tag{B.5}
$$

We recall relation (B.2), then

$$
X = (-1)^{mp}\frac{\delta_R f}{\delta\underline{\varphi}} \tag{B.6}
$$

$$
X^{\dagger} = -\frac{\delta_R f}{\delta\underline{\varphi}^{\dagger}} \tag{B.7}
$$

Substituting (B.6) and (B.7) in (B.4) , we have

$$
X_f = (-1)^{mp}\int_{T[1]M}\mu\left(\frac{\delta_R f}{\delta\underline{\varphi}}\frac{\delta_L}{\delta\underline{\varphi}^{\dagger}} - (-1)^{mp}\frac{\delta_R f}{\delta\underline{\varphi}^{\dagger}}\frac{\delta_L}{\delta\underline{\varphi}}\right) \tag{B.8}
$$

In order to obtain the BV bracket in the usual form, we apply (B.8) to a functional g, then

$$
X_f(g) = \{f,g\} = (-1)^{mp}\int_{T[1]M}\mu\left(\frac{\delta_R f}{\delta\underline{\varphi}}\frac{\delta_L g}{\delta\underline{\varphi}^{\dagger}} - (-1)^{mp}\frac{\delta_R f}{\delta\underline{\varphi}^{\dagger}}\frac{\delta_L g}{\delta\underline{\varphi}}\right) \tag{B.9}
$$

The BV bracket (B.9) enjoys the following properties:

-Antisymmetry

$$
\{f,g\} + (-1)^{(|f|+1)(|g|+1)}\{g,f\} = 0 \tag{B.10}
$$

- Jacobi identity

$$
\{f,\{g,h\}\} - \{\{f,g\},h\} - (-1)^{(|f|+1)(|g|+1)}\{g,\{f,h\}\} = 0 \tag{B.11}
$$

-Leibniz rules

$$
\{f,gh\} = \{f,g\}h + (-1)^{(|f|+1)|g|}g\{f,h\} \tag{B.12}
$$

$$
\{fg,h\} = f\{g,h\} + (-1)^{|g|(|h|+1)}\{f,h\}g \tag{B.13}
$$

B.2 BV bracket in field components

In this section we provide an expression for the BV bracket in field components. To obtain this result we can begin considering the following symplectic form

$$\Omega_{BV} = \int_M \sum_i \left(\hat{\delta}\varphi_i^\dagger \hat{\delta}\varphi_i \right), \tag{B.14}$$

where M is a $m-$dimensional manifold. We denoted with $\hat{\delta}$ the functional de Rahm operator. The field and antifield components denoted with $\varphi_i^\dagger$ and φ_i have a ghost bidegree (p_i, g_i) and $(m - p_i, -1 - g_i)$ respectively.
Given a generci functional f, we have:

$$\begin{aligned} \delta f &= \int_M \left(\sum_i \hat{\delta}\varphi_i \frac{\delta_L f}{\delta \varphi_i} + \sum_i \hat{\delta}\varphi_i^\dagger \frac{\delta_L f}{\delta \varphi_i^\dagger} \right) = \\ &= \int_M \left(\sum_i \frac{\delta_R f}{\delta \varphi_i} \hat{\delta}\varphi_i + \sum_i \frac{\delta_R f}{\delta \varphi_i^\dagger} \hat{\delta}\varphi_i^\dagger \right) \end{aligned} \tag{B.15}$$

We consider now the inner contraction i_{X_f} whose has the following form

$$i_{X_f} = \int_M \left[\sum_i X_i \frac{\delta_L}{\delta \hat{\delta} \varphi_i^\dagger} + \sum_i X_i^\dagger \frac{\delta_L}{\delta \hat{\delta} \varphi_i} \right], \tag{B.16}$$

where X_f is the Hamiltonian vector field, which has the form

$$X_f = \sum_i X_i \frac{\delta_L}{\delta \varphi_i^\dagger} + \sum_i X_i^\dagger \frac{\delta_L}{\delta \varphi_i} \tag{B.17}$$

Now we apply the inner contraction (B.16) to the BV symplectic form (B.14), then

$$\begin{aligned} i_{X_f}\Omega_{BV} &= i_{X_f} \int_M \sum_i \left(\hat{\delta}\varphi_i^\dagger \hat{\delta}\varphi_i \right) = \\ &= \int_M \sum_i \left(i_{X_f} \hat{\delta}\varphi_i^\dagger \hat{\delta}\varphi_i - (-1)^{(|f|+1)(m-p_i-g_i-1)} \hat{\delta}\varphi_i^\dagger i_{X_f} \hat{\delta}\varphi_i \right) \end{aligned}$$

We can use equation (B.16) in the previous expression, therefore

$$\begin{aligned} &= \int_M \sum_i \left(X_i \hat{\delta}\varphi^i - (-1)^{(|f|+1)(m-p_i-g_i-1)} \hat{\delta}\varphi_i^\dagger X_i^\dagger \right) = \\ &= \int_M \sum_i \left(X_i \hat{\delta}\varphi^i - (-1)^{(|f|+1)(m-p_i-g_i-1)+(m-p_i-g_i-1)(-m+p_i+g_i+|f|+1)} X_i^\dagger \hat{\delta}\varphi^{i\dagger} \right) = \\ &= \int_M \sum_i \left(X_i \hat{\delta}\varphi^i - X_i^\dagger \hat{\delta}\varphi^{i\dagger} \right) \end{aligned}$$

We recall now the following well-known relation

$$i_{X_f}\Omega_{BV} = \hat{\delta} f \tag{B.18}$$

Using (B.15), we have

$$X_i = \frac{\delta_R f}{\delta \varphi_i} \tag{B.19}$$

$$X_i^\dagger = \frac{\delta_R f}{\delta \varphi_i^\dagger} \tag{B.20}$$

Substituting (B.19) and (B.20) in (B.17), we obtain

$$X_f = \int_M \sum_i \left(\frac{\delta_R f}{\delta \varphi_i} \frac{\delta_L}{\delta \varphi_i^\dagger} - \frac{\delta_R f}{\delta \varphi_i^\dagger} \frac{\delta_L}{\delta \varphi_i} \right) \tag{B.21}$$

In order to obtain the ordinary expression for the BV bracket, we apply (B.21) to a given functional g. Finally, we get

$$X_f(g) = \{f, g\} = \int_M \sum_i \left(\frac{\delta_R f}{\delta \varphi_i} \frac{\delta_L g}{\delta \varphi_i^\dagger} - \frac{\delta_R f}{\delta \varphi_i^\dagger} \frac{\delta_L g}{\delta \varphi_i} \right) \tag{B.22}$$

Poisson bracket in field components (B.22) enjoys the properties (B.10), (B.11),(B.12) and (B.13).

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